# Functions of a Complex Variable

By

Dr. Pragya Mishra

Maharana Pratap Govt. P. G. College,Hardoi

# Cauchy-Riemann conditions

## Complex algebra

**Complex number:** $z = x + iy$ (both $x$ and $y$ are real, $i = \sqrt{-1}$.)

**Complex algebra:**

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \qquad \text{(Anologous to 2d vectors.)}$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \quad (\Rightarrow cz = c(x+iy) = cx + icy) \quad (\Rightarrow z_1 - z_2)$$

Complex conjugation: $z^* = (x+iy)^* = x - iy$

$$\Rightarrow zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation: $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Modulus (magnitude): $|z| = \sqrt{zz^*} = r = \sqrt{x^2 + y^2} \qquad \Rightarrow |z_1 z_2| = |z_1||z_2|$

Argument (phase): $\arg(z) = \theta = \arctan\left(\frac{y}{x}\right)$ ($+\pi$ if $z$ is in the 2nd or 3rd quadrants.)

$$\Rightarrow \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

## Functions of a complex variable:

All elementary functions of real variables may be extended into the complex plane.

$$\text{Example}: \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \to \quad e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

A complex function can be resolved into its *real part* and *imaginary part*:

$$f(z) = u(x, y) + iv(x, y)$$

$$\text{Examples}: \; z^2 = (x + iy)^2 = (x^2 + y^2) + i2xy$$

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$$

## Multi-valued functions and branch cuts:

$$\text{Example 1}: \; \ln z = \ln(re^{i\theta}) = \ln[re^{i(\theta + 2n\pi)}] = \ln r + i(\theta + 2n\pi) = u + iv$$

To remove the ambiguity, we can limit all phases to $(-\pi, \pi)$.
$\theta = -\pi$ is the *branch cut*.
$\ln z$ with $n = 0$ is the *principle value*.

$$\text{Example 2}: \; z^{1/2} = (re^{i\theta})^{1/2} = \left[re^{i(\theta + 2n\pi)}\right]^{1/2} = r^{1/2} e^{i(\theta + 2n\pi)/2}$$

We can let $z$ move on 2 *Riemann sheets* so that $f(z) = (re^{i\theta})^{1/2}$ is single valued everywhere.

## Cauchy-Riemann conditions

**Analytic functions**: If $f(z)$ is differentiable at $z = z_0$ and within the neighborhood of $z=z_0$, $f(z)$ is said to be **analytic** at $z = z_0$. A function that is analytic in the whole complex plane is called an *entire function*.

**Cauchy-Riemann conditions for differentiability**

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z}$$

In order to let $f$ be differentiable, $f'(z)$ must be the same in any direction of $\Delta z$.

Particularly , it is necessary that

For $\Delta z = \Delta x$, $\quad f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$

For $\Delta z = i\Delta y$, $\quad f'(z) = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$

Equating them we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ ← Cauchy-Riemann conditions

Conversely, if the Cauchy-Riemann conditions are satisfied, $f(z)$ is differentiable:

$$\frac{df}{dz}=\lim_{\Delta z\to 0}\frac{\Delta f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x+i\Delta y}=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y}{\Delta x+i\Delta y}$$

$$=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)}{\Delta x+i\Delta y}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\quad \text{and}=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right).$$

## More about Cauchy-Riemann conditions:

1) It is a very strong restraint to functions of a complex variable.

2) $\frac{df}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial(iy)}+i\frac{\partial v}{\partial(iy)}.$

3) $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=0\Rightarrow\nabla u\cdot\nabla v=0\Rightarrow\nabla u\perp\nabla v\Rightarrow u=c_1\perp v=c_2$

4) Equivalent to $\frac{\partial f}{\partial z^*}=0$, so that $f(z,z^*)$ only depends on $z$:

$$\frac{\partial f}{\partial z^*}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}=\frac{\partial f}{\partial x}\frac{1}{2}+\frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)=0\Rightarrow\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=0\Rightarrow\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)=0\Rightarrow\cdots$$

e.g., $f=x-iy$ is everywhere continuous but not analytic.

## Reading: General search for Cauchy-Riemann conditions:

Our Cauchy-Riemann conditions were derived by requiring $f'(z)$ be the same when $z$ changes along $x$ or $y$ directions. How about other directions?

Here I do a general search for the conditions of differentiability.

$$f'(z)=\frac{df}{dz}=\frac{du+idv}{dx+idy}=\frac{\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)+i\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)}{dx+idy}=\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}\right)+i\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}\right)}{1+i\frac{dy}{dx}}$$

Now let $\frac{dy}{dx}=p$, the direction of the change of $z$. We want to find the condition under which $f'(z)$ does not depend on $p$.

$$\frac{df'(z)}{dp}=0=\frac{d}{dp}\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}p\right)+i\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}p\right)}{1+ip}=\frac{\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)(1+ip)-i\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}p\right)+\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}p\right)}{(1+ip)^2}$$

$$=\frac{\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)}{(1+ip)^2}\Rightarrow\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$$

That is, if we require $f'(z)$ be the same at all directions, we get the same Cauchy - Riemann conditions.

# Cauchy's theorem

## Cauchy's integral theorem

**Contour integral**:

$$\int_{z_1}^{z_2} f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

**Cauchy's integral theorem:** If $f(z)$ is analytic in a simply connected region $R$, [and $f'(z)$ is continuous throughout this region, ] then for any closed path $C$ in $R$, the contour integral of $f(z)$ around $C$ is zero: $\oint_C f(z)dz = 0$

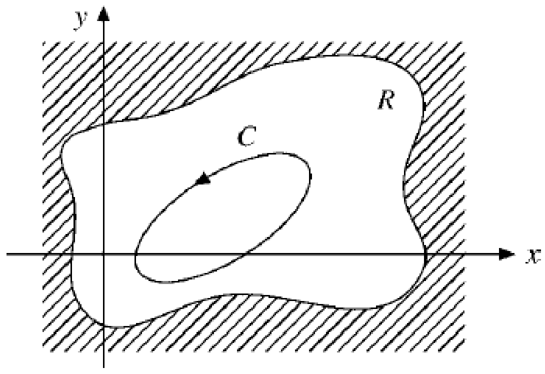
Proof using Stokes' theorem: $\oint_C \mathbf{V}\cdot d\boldsymbol{\lambda} = \iint_S \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}$

$$\oint_C \left(V_x dx + V_y dy\right) = \iint_S \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy$$

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

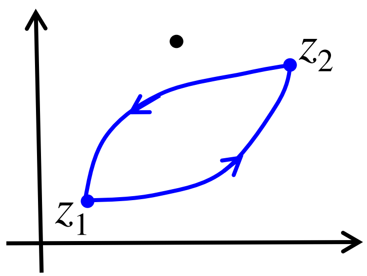
$$= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

$$= 0$$



Cauchy-Goursat proof: The continuity of $f'(z)$ is not necessary.

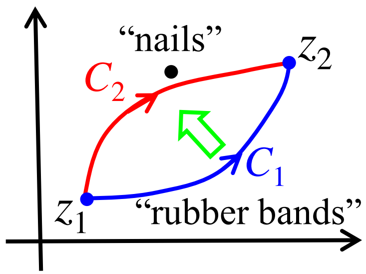
Corollary: An open contour integral for an analytic function is independent of the path, if there is no singular points between the paths.




$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)dz$$

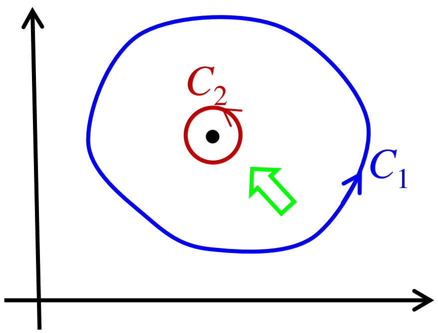
**Contour deformation theorem:**

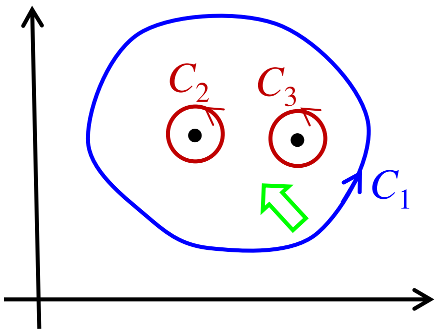
A contour of a complex integral can be arbitrarily deformed through an analytic region without changing the integral.




1) It applies to both open and closed contours.
2) One can even split closed contours.
Proof: Deform the contour bit by bit.

Examples:
1) Cauchy's integral theorem.
(Let the contour shrink to a point.)
2) Cauchy's integral formula.
(Let the contour shrink to a small circle.)



# Cauchy's integral formula

**Cauchy's integral formula:**

If $f(z)$ is analytic within and on a closed contour $C$, then for any point $z_0$ within $C$,

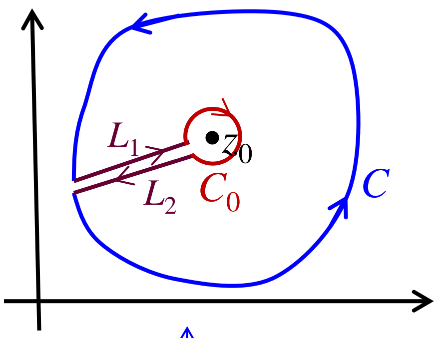
$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

Proof :

$$\oint_C \frac{f(z)}{z-z_0}dz + \oint_{L_1} \frac{f(z)}{z-z_0}dz + \oint_{C_0} \frac{f(z)}{z-z_0}dz + \oint_{L_2} \frac{f(z)}{z-z_0}dz = 0$$

$$\oint_C \frac{f(z)}{z-z_0}dz = -\oint_{C_0} \frac{f(z)}{z-z_0}dz = -\int_{2\pi}^{0} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} rie^{i\theta}d\theta \quad (\text{Let } r \to 0)$$

$$= 2\pi i f(z_0)$$




Can directly use the contour deformation theorem.

**Derivatives** of $f(z)$: $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

Corollary: If a function is analytic, then its derivatives of all orders exist.
Corollary: If a function is analytic, then it can be expanded in Taylor series.

**Cauchy's inequality:** If $f(z) = \sum a_n z^n$ is analytic and bounded, $\left|f(z)\right|_{|z|=r} \le M$, then $|a_n| r^n \le M$. (That is, $a_n$ is bounded.)

$$\text{Proof}: f^{(n)}(0) = n! a_n = \frac{n!}{2\pi i}\oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz \Rightarrow |a_n| = \frac{1}{2\pi}\left|\oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz\right| \le \frac{M}{r^n} \Rightarrow |a_n| r^n \le M$$

**Liouville's theorem:** If a function is analytic and bounded in the entire complex plane, then this function is a constant.

Proof : $|a_n| \le \frac{M}{r^n}$, let $r \to \infty$, then $a_n = 0$ for $n > 0$. $f(z) = a_0$.

**Fundamental theorem of algebra:** $P(z) = \sum_{i=0}^{n} a_i z^i \ (n > 0, a_n \ne 0)$ has $n$ roots.

Suppose $P(z)$ has no roots, then $1/P(z)$ is analytic and bounded as $|z| \to \infty$. Then $P(z)$ is constant. That is nonsense. Therefore $P(z)$ has at least one root we can divide out.

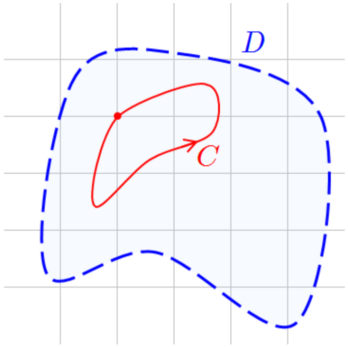
**Morera's theorem:** If $f(z)$ is continuous and $\oint_C f(z)dz = 0$ for every closed contour within a simply connected region, then $f(z)$ is analytic in this region.

Proof :

$$\oint_C f(z)dz = 0 \Rightarrow \int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) \Rightarrow F'(z) = f(z)$$

$\Rightarrow F(z)$ is analytic

$\Rightarrow F'(z) = f(z)$ is analytic

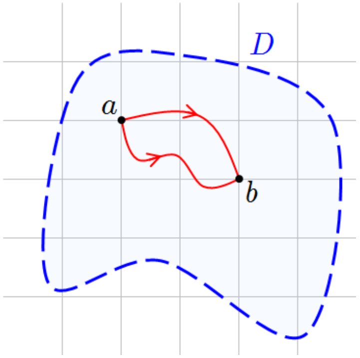



Why $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$?

Let $\int_{z_1}^{z_2} f(z)dz = G(z_1, z_2)$, then

$$G(z_1, z_2) = G(z_1, 0) + G(0, z_2)$$

$$= -G(0, z_1) + G(0, z_2) = -F(z_1) + F(z_2)$$



# Analytic continuation

## Laurent expansion

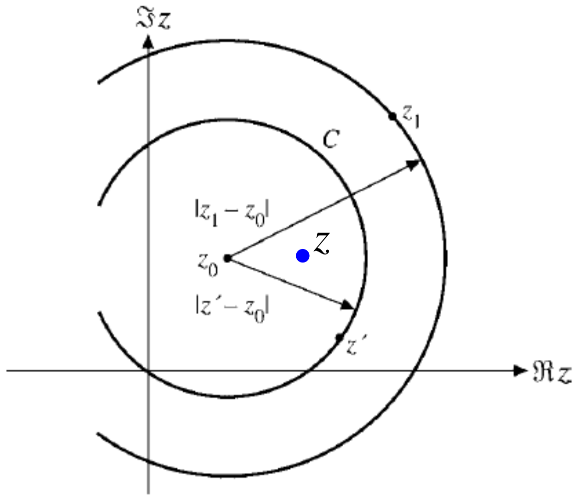
Taylor expansion for functions of a complex variable:
Expanding an analytic function $f(z)$ about $z = z_0$, where $z_1$ is the nearest singular point.

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}dz' = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)-(z-z_0)}dz'$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}dz' = \frac{1}{2\pi i}\oint_C \frac{\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n f(z')}{(z'-z_0)}dz'$$

$$= \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty}\frac{(z-z_0)^n f(z')}{(z'-z_0)^{n+1}}dz' = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

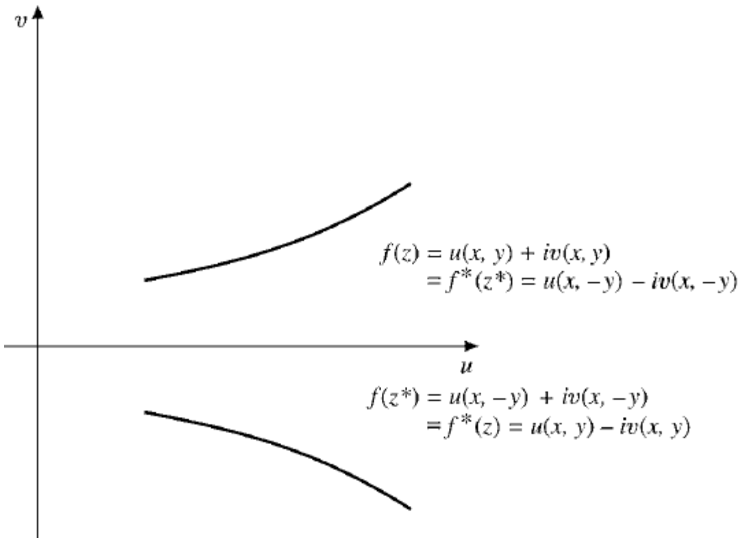


## Schwarz's reflection principle:

If $f(z)$ is 1) analytic over a region including the real axis, and 2) real when $z$ is real, then $f^*(z) = f(z^*)$.

Proof : $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(z - x_0)^n$

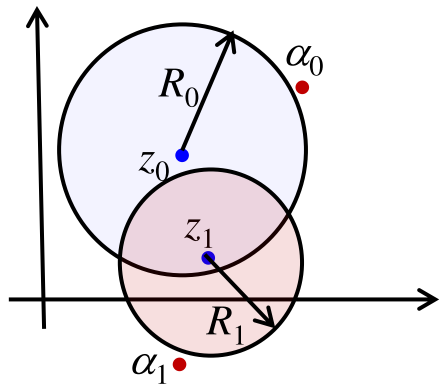
$$\Rightarrow f^*(z) = f(z^*)$$




Examples: most of the elementary functions.

**Analytic continuation:** Suppose $f(z)$ is analytic around $z = z_0$, we can expand it about $z = z_0$ in a Taylor series:

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m$$




This series converges inside a circle with a radius of convergence $R_0 = |\alpha_0 - z_0|$, where $\alpha_0$ is the nearest singularity from $z = z_0$.

We can also expand $f(z)$ about another point $z = z_1$ within the circle $R_0$: $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z - z_1)^n$ .

In general, the new circle has a radius of convergence $R_1 = |\alpha_1 - z_1|$ and contains points not within the first circle.

From the first expansion, $f^{(n)}(z_1) = \sum_{m=n}^{\infty} \frac{f^{(m)}(z_0)}{(m-n)!}(z_1 - z_0)^{m-n}$

Plug into the second expansion, $f(z) = \sum_{n=0, m=n}^{\infty} \frac{f^{(m)}(z_0)(z_1 - z_0)^{m-n}}{n!(m-n)!}(z - z_1)^n$

Consequences:

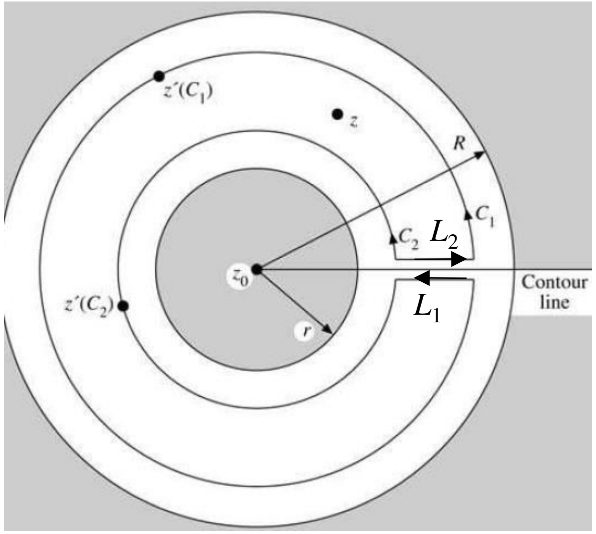
1) $f(z)$ can be analytically continued over the complex plane, excluding singularities.

2) If $f(z)$ is analytic, its values at one region determines its values everywhere.

# Laurent expansion

## Laurent expansion

Problem: Expanding a function $f(z)$ that is analytic in an annular region (between $r$ and $R$).




$$f(z) = \frac{1}{2\pi i}\oint_{C_1+L_1+\tilde{C}_2+L_2} \frac{f(z')dz'}{z'-z}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{z'-z} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{z'-z}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)-(z-z_0)} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)\left(1-\dfrac{z-z_0}{z'-z_0}\right)} + \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{(z-z_0)\left(1-\dfrac{z'-z_0}{z-z_0}\right)}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{1}{(z-z_0)^{m+1}}\oint_{C_2}(z'-z_0)^m f(z')dz'$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{1}{(z-z_0)^{m}}\oint_{C_2}(z'-z_0)^{m-1} f(z')dz'$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=-1}^{-\infty}(z-z_0)^n\oint_{C_2}\frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

$$= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}(z-z_0)^n\oint_{C}\frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

← $C$ is any contour that encloses $z_0$ and lies between $r$ and $R$ (*deformation theorem*).

**Laurent expansion**:

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n, \quad a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

1) Singular points of the integrand.
   For $n < 0$, the singular points are determined by $f(z)$. For $n \geq 0$, the singular points are determined by both $f(z)$ and $1/(z'-z_0)^{n+1}$.
2) If $f(z)$ is *analytic* inside $C$, then the Laurent series reduces to a Taylor series:

$$a_n=\begin{cases}\dfrac{f^{(n)}(z_0)}{n!}, & n\geq 0,\\ 0, & n<0.\end{cases}$$

3) Although $a_n$ has a general contour integral form, In most times we need to use straight forward complex algebra to find $a_n$.

## Laurent expansion: Examples

Example 1: Expand $f(z)=\dfrac{z^3}{(z-1)^2}$ about $z_0$=1.

$$\frac{z^3}{(z-1)^2}=\frac{[(z-1)+1]^3}{(z-1)^2}=\frac{(z-1)^3+3(z-1)^2+3(z-1)+1}{(z-1)^2}=\frac{1}{(z-1)^2}+\frac{3}{z-1}+3+(z-1)$$

Example 2: Expand $f(z)=\dfrac{1}{z^2+1}$ about $z_0$=$i$.

$$f(z)=\frac{1}{z^2+1}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{2i+z-i}\right)$$

$$=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{2i}\cdot\frac{1}{1+\frac{z-i}{2i}}\right)=\frac{1}{2i}\frac{1}{z-i}-\frac{1}{(2i)^2}\sum_{n=0}^{\infty}\left(-\frac{1}{2i}\right)^n(z-i)^n$$

$$=-\frac{i}{2}\frac{1}{z-i}+\frac{1}{4}+\frac{i}{8}(z-i)+\cdots$$

# Branch points and branch cuts

## Singularities

**Poles**: In a Laurent expansion $f(z)=\sum_{m=-\infty}^{\infty} a_m (z-z_0)^m$, if $a_m=0$ for $m<-n<0$ and $a_{-n}\neq 0$,

then $z_0$ is said to be a *pole of order n*.
A pole of order 1 is called a *simple pole*.
A pole of infinite order (when expanded about $z_0$) is called an *essential singularity*.

The behavior of a function $f(z)$ at infinity is defined using the behavior of $f(1/t)$ at $t=0$.

Examples:

$$1)\ \frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=\frac{1}{2i}\left[-\frac{1}{z+i}-\frac{1}{2i-(z+i)}\right]=-\frac{1}{2i}\frac{1}{z+i}+\frac{1}{4}\frac{1}{1-(z+i)/2i}$$

$$=-\frac{1}{2i}\frac{1}{z+i}+\frac{1}{4}\left[1+\frac{z+i}{2i}+\left(\frac{z+i}{2i}\right)^2+\cdots\right]$$ has a single pole at $z=-i$.

$$2)\ \sin z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!},\quad \sin\frac{1}{t}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{t^{2n+1}}$$

$\sin z$ thus has an essential singularity at infinity.

3) $z^2+1$ has a pole of order 2 at infinity.

**Branch points and branch cuts**:

**Branch point**: A point $z_0$ around which a function $f(z)$ is discontinuous after going a small circuit. E.g., $z_0 = 1$ for $\sqrt{z\text{-}1}$, $z_0 = 0$ for $\ln z$.

**Branch cut**: A curve drawn in the complex plane such that if a path is not allowed to cross this curve, a multi-valued function along the path will be single valued.
Branch cuts are *usually* taken between pairs of branch points. E.g., for $\sqrt{z\text{-}1}$ , the curve connects $z$=1 and $z$ = ◎ can serve as a branch cut.

Examples of branch points and branch cuts:

1. $f(z) = z^a = r^a(\cos a\theta + i\sin a\theta)$

If $a$ is a rational number, $a = p/q$, then circling the branch point $z = 0$ $q$ times will bring $f(z)$ back to its original value. This branch point is said to be *algebraic*, and $q$ is called the order of the branch point.
If $a$ is an irrational number, there will be no number of turns that can bring $f(z)$ back to its original value. The branch point is said to be *logarithmic*.

2. $f(z)=\sqrt{(z-1)(z+1)}$

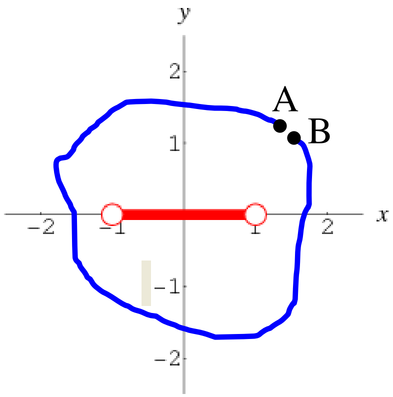
We can choose a branch cut from $z$ = -1 to $z$ = 1 (or any curve connecting these two points). The function will be single-valued, because both points will be circled.

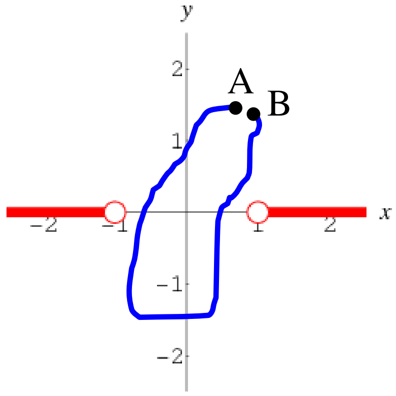
Alternatively, we can choose a branch cut which connects each branch point to infinity. The function will be single-valued, because neither points will be circled.

It is notable that these two choices result in different functions. E.g., if $f(i)=\sqrt{2}i$, then

$f(-i)=-\sqrt{2}i$ for the first choice and $f(-i)=\sqrt{2}i$

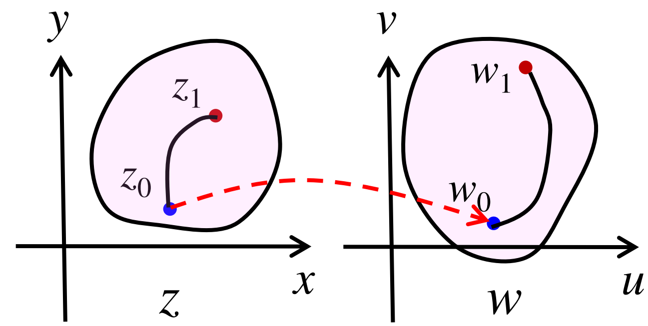
for the second choice.



## Mapping

## Mapping




**Mapping**: A complex function $w(z) = u(x, y) + iv(x, y)$ can be thought of as describing a mapping from the complex $z$-plane into the complex $w$-plane.
In general, a point in the $z$-plane is mapped into a point in the $w$-plane. A curve in the $z$-plane is mapped into a curve in the $w$-plane. An area in the $z$-plane is mapped into an area in the $w$-plane.

**Examples of mapping**:

Translation:

$$w = z + z_0$$

Rotation:

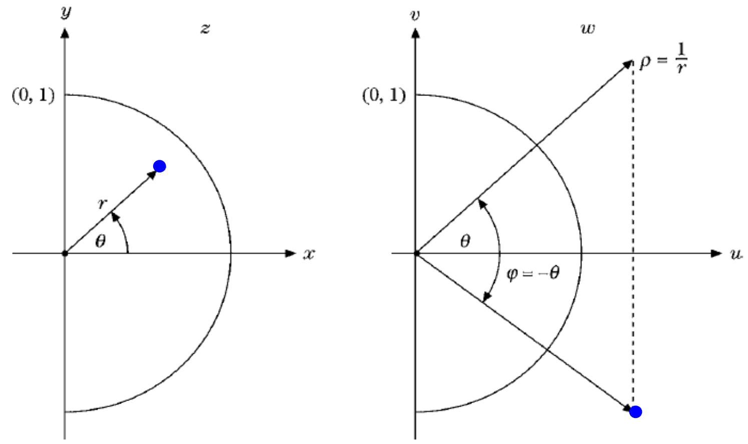
$w = zz_0$, or

$$\rho e^{i\varphi} = re^{i\theta} \cdot r_0 e^{i\theta_0} \Rightarrow \begin{cases} \rho = r \cdot r_0 \\ \varphi = \theta + \theta_0 \end{cases}$$

## Inversion:

$$w = \frac{1}{z}, \text{ or}$$

$$\rho e^{i\varphi} = \frac{1}{re^{i\theta}} \Rightarrow \begin{cases} \rho = \dfrac{1}{r} \\ \varphi = -\theta \end{cases}$$



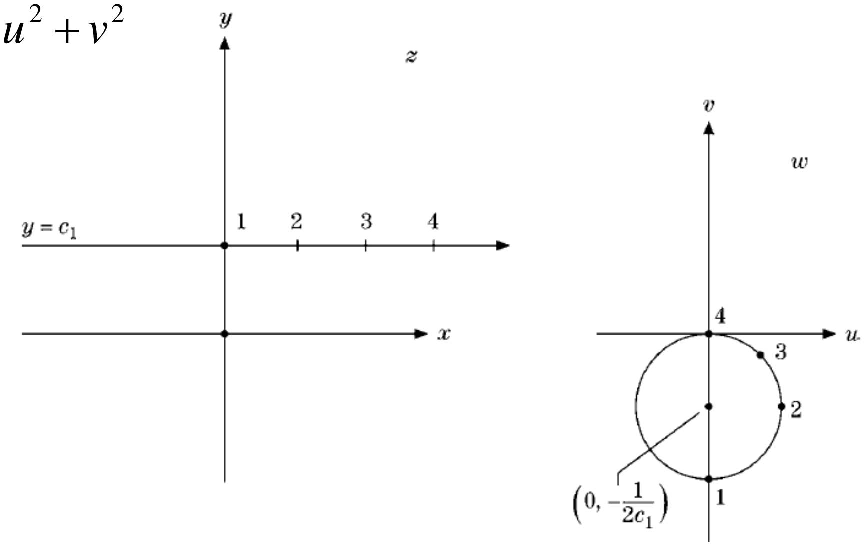

In Cartesian coordinates:

$$w = \frac{1}{z} \Rightarrow u + iv = \frac{1}{x+iy} \Rightarrow \begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = -\dfrac{y}{x^2+y^2} \end{cases}, \begin{cases} x = \dfrac{u}{u^2+v^2} \\ y = -\dfrac{v}{u^2+v^2} \end{cases}.$$

A straight line is mapped into a circle:

$$y = ax + b \Rightarrow -\frac{v}{u^2+v^2} = \frac{au}{u^2+v^2} + b$$

$$\Rightarrow b(u^2+v^2) + au + v = 0.$$



# Conformal mapping

**Conformal mapping**: The function $w(z)$ is said to be conformal at $z_0$ if it preserves the angle between any two curves through $z_0$.

If $w(z)$ is analytic and $w'(z_0) \neq 0$, then $w(z)$ is conformal at $z_0$.

Proof: Since $w(z)$ is analytic and $w'(z_0) \neq 0$, we can expand $w(z)$ around $z = z_0$ in a Taylor series:

$$w = w(z_0) + w'(z_0)(z - z_0) + \frac{1}{2} w''(z_0)(z - z_0)^2 + \cdots$$

$$\lim_{z - z_0 \to 0} \frac{w - w_0}{z - z_0} = w'(z_0), \text{ or } w - w_0 \approx w'(z_0)(z - z_0).$$

$$w - w_0 = Ae^{i\alpha}(z - z_0) \Rightarrow \varphi = \alpha + \theta \Rightarrow \varphi_2 - \varphi_1 = \theta_2 - \theta_1.$$

1) At any point where $w(z)$ is conformal, the mapping consists of a rotation and a dilation.
2) The local amount of rotation and dilation varies from point to point. Therefore a straight line is usually mapped into a curve.
3) A curvilinear orthogonal coordinate system is mapped to another curvilinear orthogonal coordinate system .

What happens if $w'(z_0) = 0$?

Suppose $w^{(n)}(z_0)$ is the first non-vanishing derivative at $z_0$.

$$w - w_0 \approx \frac{w^{(n)}(z_0)}{n!}(z - z_0)^n \Rightarrow \rho e^{i\varphi} = \frac{1}{n!} Be^{i\beta}(re^{i\theta})^n \Rightarrow \begin{cases} \rho = \dfrac{Br^n}{n!} \\ \varphi = n\theta + \beta \end{cases}$$

This means that at $z = z_0$ the angle between any two curves is magnified by a factor $n$ and then rotated by $\beta$.