A **Simple** Introduction to

Integral EQUATIONS

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Some Integral Equation Examples

$$y(t) = \int_0^t \frac{2s - y(s)}{1 - y(s)}\, ds$$

Volterra **second** kind

$$y(t) = \sqrt{t} - \int_0^t 2\sqrt{ts}\, y(s)\, ds$$

Volterra **second** kind

$$y(t) = \sqrt{t} - \int_0^1 2\sqrt{ts}\, y(s)\, ds$$

Fredholm **second** kind

$$\int_0^t \frac{y(s)\, ds}{\sqrt{t^2 - s^2}} = t$$

Volterra **first** kind

Integral Equations from Differential Equations

First-Order Initial Value Problem $\Rightarrow$ **Volterra** Integral Equation Of the **Second Kind**

$$y'(t) = F(t) + g(t, y(t)), \quad y(a) = y_0$$

$$y(t) - y_0 = \int_a^t F(s)\,ds + \int_a^t g(s, y(s))\,ds$$

Used to **prove theorems** about differential equations.

Used to **derive numerical methods** for differential equations.

- Volterra IEs of the second kind (VESKs)
- Symbolic solution of separable VESKs
- Volterra sequences and iterative solutions
- Theory of linear VESKs
- Brief mention of Fredholm IEs of 2^{nd} kind
- An age-structured population model

Volterra Integral Equations of the Second Kind (VESKs)

Given $g : \mathbf{R}^3 \rightarrow \mathbf{R}$ and $f: \mathbf{R} \rightarrow \mathbf{R}$, **find** $y: \mathbf{R} \rightarrow \mathbf{R}$ such that

$$y(t) = f(t) + \int_a^t g(t, s, y(s))\, ds$$

[equivalent to an initial value problem if g is independent of t]

$$y(t) - y_0 = \int_a^t F(s)\, ds + \int_a^t g(s, y(s))\, ds$$

Volterra Integral Equations of the Second Kind (VESKs)

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$$y(t) = f(t) + \int_a^t g(t, s, y(s))\, ds$$

Linear:

$$y(t) = f(t) + \int_a^t k(t, s)\, y(s)\, ds$$

Separable:

$$k(t, s) = p(t)\, q(s)$$

Convolved:

$$k(t, s) = r(t - s)$$

Symbolic Solution of Separable VESKs

Solve $$y(t) = \sqrt{t} - \int_0^t 2\sqrt{ts}\, y(s)\, ds$$

Let $$Y(t) = \int_0^t \sqrt{s}\, y(s)\, ds$$

Then $$y(t) = \sqrt{t} - 2\sqrt{t}\, Y(t)$$

New Problem:

$$Y'(t) = \sqrt{t}\, y = t - 2t\, Y(t), \quad Y(0) = 0$$

Solution: $$y(t) = \sqrt{t}\, e^{-t^2}$$

Solve $$y(t) = 1 - \int_0^t (t - s)\, y(s)\, ds$$

Let $$Y(t) = \int_0^t y(s)\, ds$$

and $$Z(t) = \int_0^t s\, y(s)\, ds$$

New Problem:

$$Y'' + Y = 0, \quad Y(0) = 0, \quad Y'(0) = 1,$$

$$Z = Y' + tY - 1$$

Solution: $$y(t) = \cos t$$

Volterra Sequences

Given f, g, and y_0, **define** $y_1, y_2, \ldots$ by

$$y_1(t) = f(t) + \int_a^t g(t, s, y_0(s))\, ds$$

$$y_2(t) = f(t) + \int_a^t g(t, s, y_1(s))\, ds$$

and so on.

Volterra Sequences

Given f, g, and y_0, **define** y_1, y_2, **…** by

$$y_n(t) = f(t) + \int_a^t g(t, s, y_{n-1}(s))\,ds, \quad n = 1, 2, \ldots$$

- Does y_n **converge** to some y **?**
- If so, does y **solve** the VESK**?**

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- Does y_n **converge** to some y **?**
- If so, does y **solve** the VESK**?**
- If so, is this a **useful** iterative method**?**

A Linear Example

$$y(t) = 1 - \int_0^t (t - s)\, y(s)\, ds$$

Let $\mathbf{y_0 = 1}$. **Then**

$$y_1(t) = 1 - \int_0^t (t - s)\, ds = \cdots = 1 - \tfrac{1}{2} t^2$$

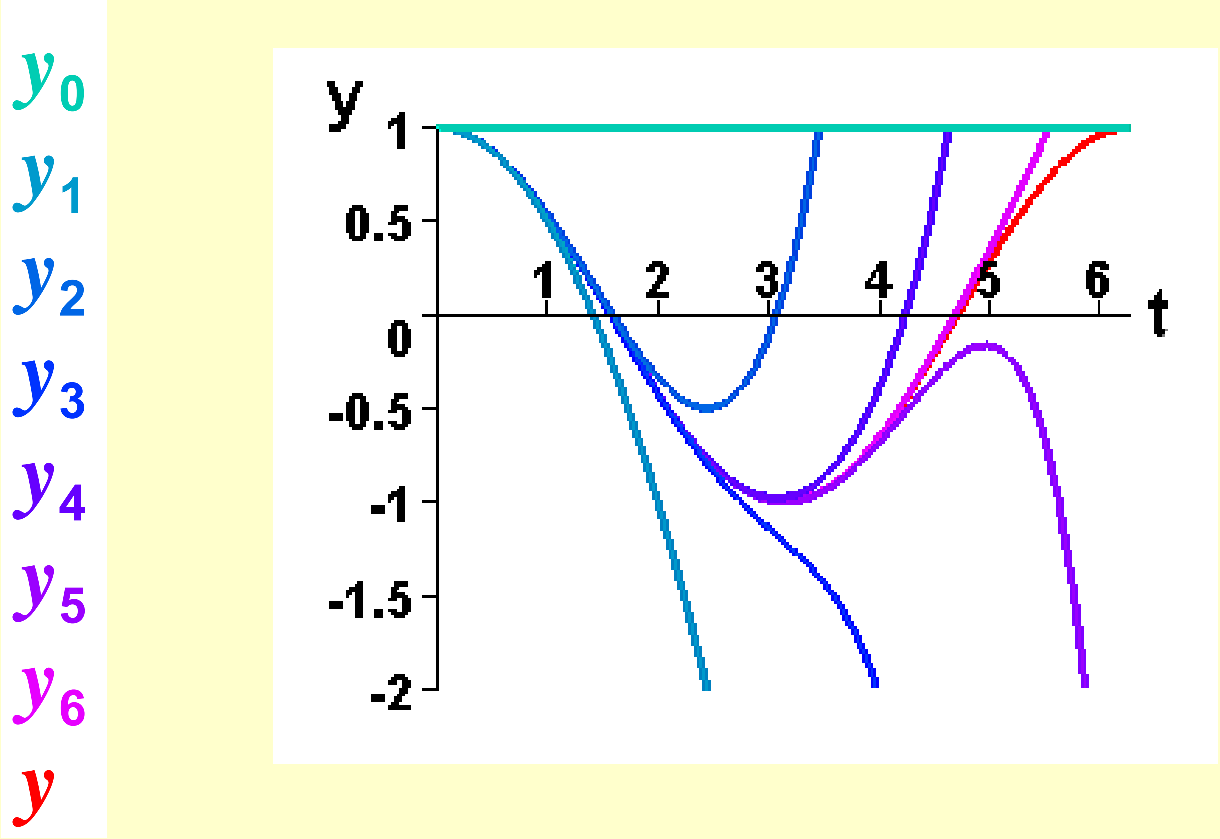
$$y_2(t) = 1 - \int_0^t (t - s)(1 - \tfrac{1}{2} s^2)\, ds = \cdots = 1 - \tfrac{1}{2} t^2 + \tfrac{1}{24} t^4$$

$$\downarrow\downarrow$$

$$y_\infty = 1 - \tfrac{1}{2} t^2 + \tfrac{1}{24} t^4 - \tfrac{1}{6!} t^6 + \cdots = \cos t$$

The sequence **converges** to the known **solution**.

Convergence of the Sequence



A Nonlinear Example

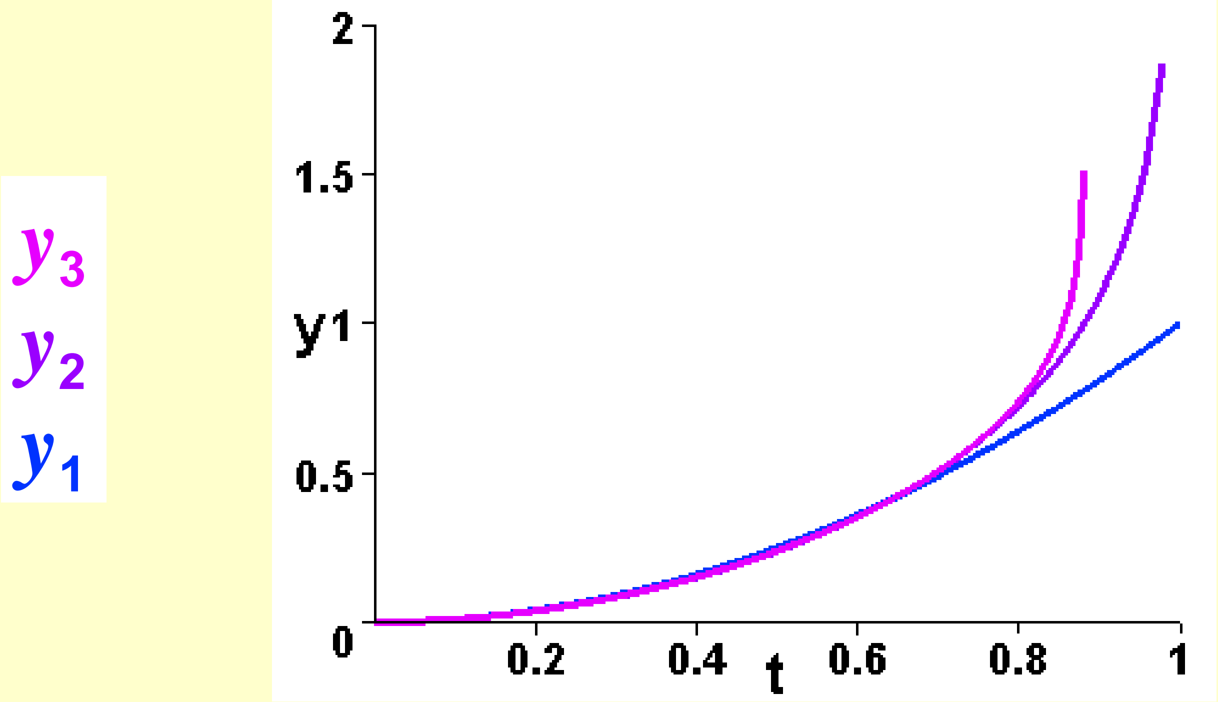
$$y(t) = \int_0^t \frac{2s - y(s)}{1 - y(s)}\,ds$$

Let $y_0 = 0$. Then $y_1(t) = \int_0^t 2s\,ds = t^2$

$$y_2(t) = \int_0^t \frac{2s - s^2}{1 - s^2}\,ds = t - \tfrac{1}{2}\ln(1-t) - \tfrac{3}{2}\ln(1+t)$$

$$y_3(t) = \int_0^t \frac{\ln(1-s) + 3\ln(1+s) + 2s}{\ln(1-s) + 3\ln(1+s) - 2s + 2}\,ds$$

Convergence of the Sequence



Lemma 1

Homogeneous linear Volterra sequences

$$y_n(t) = \int_a^t k(t,s)\, y_{n-1}(s)\, ds\,, \quad n = 1,2,\ldots$$

decay to 0 on [a, T] whenever k is bounded.

Proof: (a=0)

Choose constants K, Y and T such that

$$|k(t,s)| \le K, \quad |y_0(t)| \le Y, \quad \forall\, 0 \le t, s \le T$$

Then

$$|y_1(t)| \le \int_0^t \left| k(t,s)\, y_0(s) \right| ds \le \int_0^t KY\, ds = YKt$$

Given $|k(t,s)| \le K, \quad |y_0(t)| \le Y, \quad \forall\, 0 \le t,s \le T$

$$y_n(t) = \int_0^t k(t,s)\, y_{n-1}(s)\, ds, \quad n = 1,2,\ldots$$

we have $|y_1(t)| \le \int_0^t KY\, ds = YKt$

$$|y_2(t)| \le \int_0^t K(YKs)\, ds = \frac{1}{2} YK^2 t^2$$

$$|y_3(t)| \le \int_0^t K(\tfrac{1}{2} YK^2 s^2)\, ds = \frac{1}{3!} YK^3 t^3$$

In general,

$|y_n(t)| \le \dfrac{1}{n!} YK^n t^n \quad \forall\, n$ so $\lim\limits_{n\to\infty} y_n = 0$

Theorem 2

Homogeneous linear VESKs

$$y(t) = \int_a^t k(t,s)\, y(s)\, ds$$

have the unique solution $y \equiv 0$.

Proof:

Let φ be a **solution.**

Choose $y_0 = \varphi$.

Then $y_n = \varphi$ **for all** n, **so** $y_n \to \varphi$.

But the **lemma requires** $y_n \to 0$.

Therefore $\varphi \equiv 0$.

Theorem 3

Linear VESKs have at most one solution.

Proof:

Suppose $$\varphi(t) = f(t) + \int_a^t k(t,s)\,\varphi(s)\,ds$$

and $$\psi(t) = f(t) + \int_a^t k(t,s)\,\psi(s)\,ds$$

Let $y = \varphi - \psi$. **Then**

$$y(t) = \int_a^t k(t,s)\,y(s)\,ds$$

By Theorem 2, $y \equiv 0$**; hence,** $\varphi = \psi$.

Lemma 4

Linear Volterra sequences with $y_0 = f$ converge.

Sketch of Proof:

Define $$y_n(t) = f(t) + \int_a^t k(t,s)\, y_{n-1}(s)\, ds, \quad y_0 = f$$

$$g_n = \sum_{m=1}^{\infty} \left| y_{n+m} - y_{n+m-1} \right|$$

1) $g_n \to 0$

2) $\left| y_{n+m} - y_n \right| = \cdots < g_n$

Together, these properties **prove** the **Lemma**.

Theorem 5

Linear VESKs have a unique solution.

Proof:

By **Lemma 4**, the sequence

$$y_n(t) = f(t) + \int_a^t k(t,s)\, y_{n-1}(s)\, ds, \quad y_0 = f$$

converges to some y. Taking a **limit** as $n \to \infty$:

$$y(t) = f(t) + \int_a^t k(t,s)\, y(s)\, ds$$

The **solution** is **unique** by **Theorem 3.**

Fredholm Integral Equations of the Second Kind (linear)

Given $g: \mathbf{R}^3 \to \mathbf{R}$ and $f: \mathbf{R} \to \mathbf{R}$, **find** $y: \mathbf{R} \to \mathbf{R}$ such that

$$y(t) = f(t) + \int_a^b k(t,s)\, y(s)\, ds$$

- **Separable FESKs** can be **solved symbolically.**
- **Fredholm sequences converge** (more slowly than Volterra sequences) **only if** $\|k\|$ **is small enough.**
- The **theorems hold** if and **only if** $\|k\|$ **is small enough.**

An Age-Structured Population

Given

- An initial population of known age distribution
- Age-dependent birthrate
- Age-dependent death rate

Find

- Total birthrate
- Total population
- Age distribution

An Age-Structured Population

Simplified Version

Given

- **An initial population of *newborns***
- **Age-dependent birthrate**
- **Age-*in*dependent death rate**

Find

- **Total birthrate**
- **Total population**

The Founding Mothers

Assume a one-sex population.

Starting population is 1 unit.

Life expectancy is 1 time unit.

$$p_0' = -p_0, \quad p_0(0) = 1$$

Result:

$$p_0(t) = e^{-t}$$

Basic Birthrate Facts

Total Birthrate =

Births to Founding Mothers

+

Births to Native Daughters

$$b(t) = m(t) + d(t)$$

Let $f(t)$ be the number of **births per mother of age** t **per unit time**.

$$m(t) = f(t)\, e^{-t}$$

Births to Native Daughters

Consider daughters of ages x to $x+dx$.

All were born between $t-x-dx$ and $t-x$.

The initial number was $b(t-x)dx$.

The number at time t is $b(t-x)e^{-x}dx$.

The rate of births is $f(x)b(t-x)e^{-x}dx$

$=m(x)b(t-x)dx$.

$$d(t) = \int_0^t m(x)\,b(t-x)\,dx$$

The Renewal Equation

$$b(t) = m(t) + \int_0^t m(x)\, b(t - x)\, dx$$

or

$$b(t) = m(t) + \int_0^t m(t - x)\, b(x)\, dx$$

This is a linear VESK with convolved kernel.
It can be solved by Laplace transform.

Solution of the Renewal Equation

Given the fecundity function f, **define**

$$F(s) = L\left[f(t)\right], \quad r(t) = L^{-1}\left[\frac{F(s)}{1 - F(s)}\right]$$

where L is the **Laplace transform. Then**

$$b(t) = e^{-t} r(t)$$

$$p(t) = e^{-t} + e^{-t} \int_0^t r(x)\, dx$$

A Specific Example

Let $f(x) = axe^{-2x}$.

Then

$$b(t) = \frac{\sqrt{a}}{2} e^{(\sqrt{a}-3)t} - \frac{\sqrt{a}}{2} e^{(-\sqrt{a}-3)t}$$

$$p(t) = \frac{\sqrt{a}}{2(\sqrt{a}-2)} e^{(\sqrt{a}-3)t} - \frac{4}{a-4} e^{-t} + \frac{\sqrt{a}}{2(\sqrt{a}\_2)} e^{(-\sqrt{a}-3)t}$$

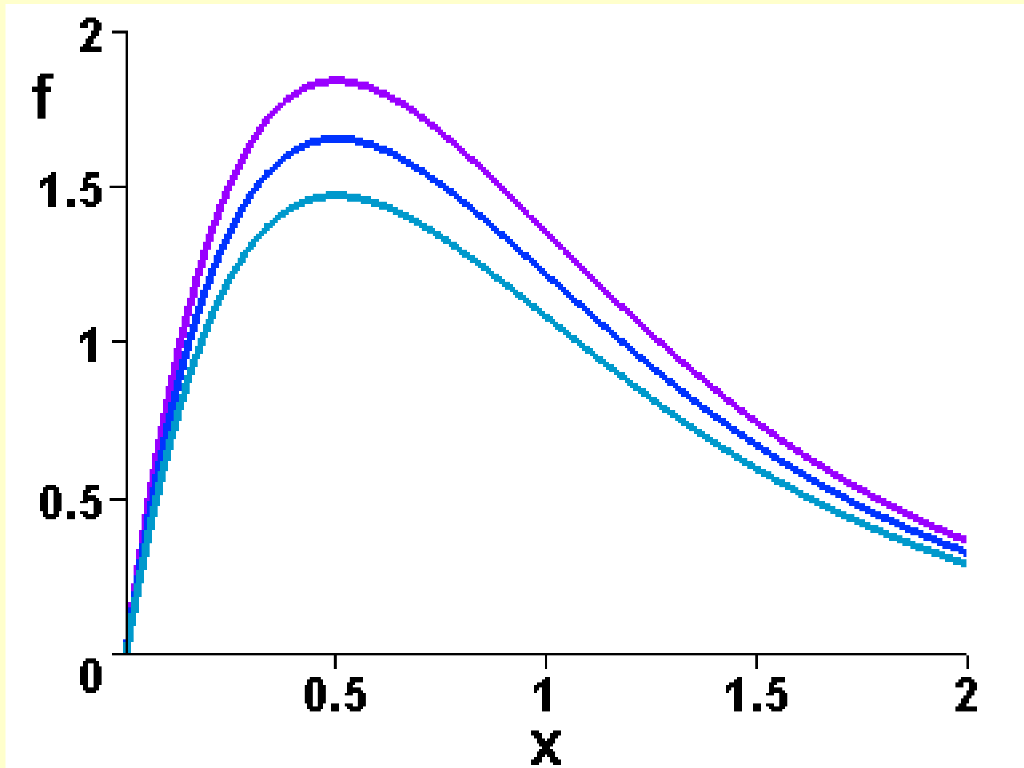
If $a=9$, then $p(t) \to 1.5$.

Exponential growth for $a>9$.

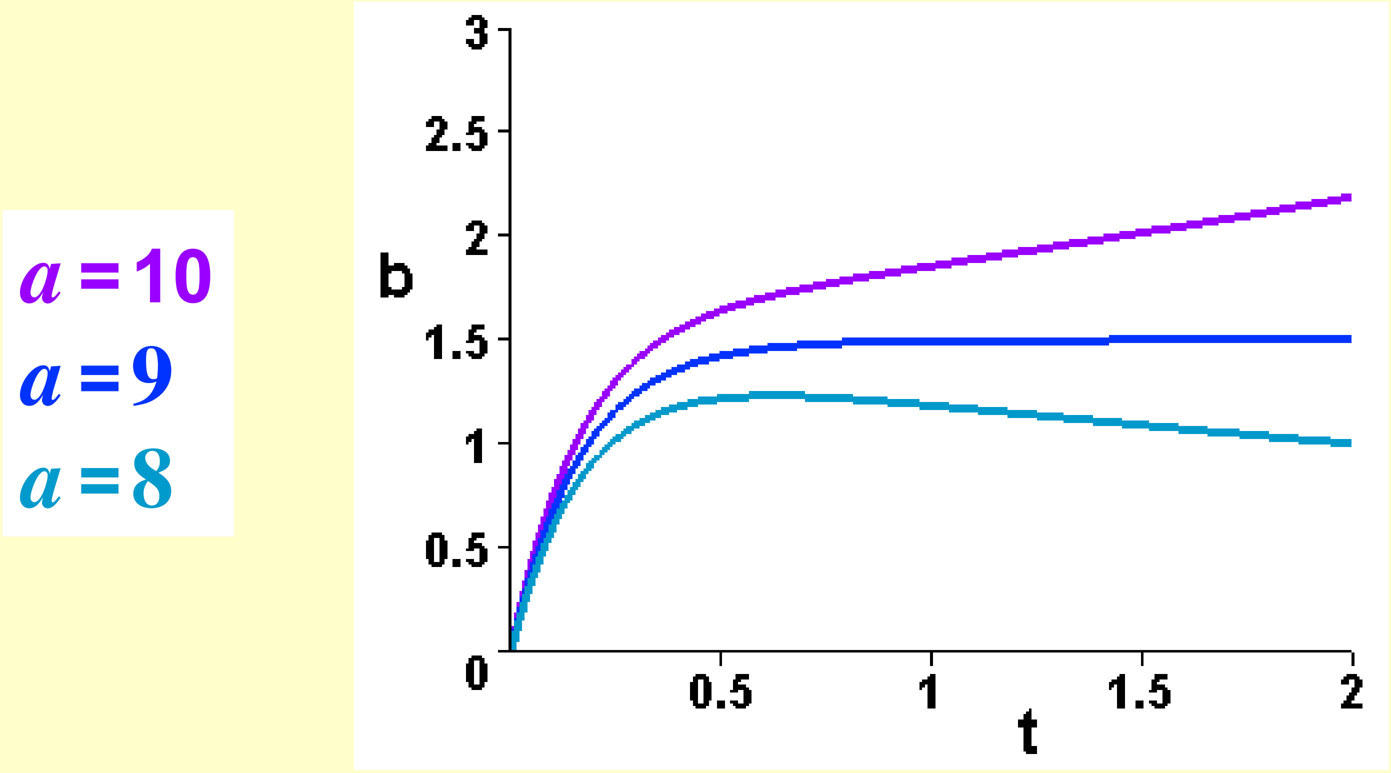
The population dies if $a<9$.

Fecundity

$a = 10$
$a = 9$
$a = 8$



Birth Rate



Population

$a = 10$
$a = 9$
$a = 8$