FLUID DYNAMICS

M.A./MSc. Mathematics (Final)

Dr. Pragya Mishra
Assistant Professor, Mathematics
Maharana Pratap Govt. P.G.Coleege, Hardoi.

Fluid Dynamics

- **Fluid dynamics** is the science treating the study of fluids in motion. By the term fluid, we mean a substance that flows i.e. which is not a solid. Fluids may be divided into two categories

  (i) liquids which are incompressible i.e. their volumes do not change when the pressure changes

  (ii) gases which are compressible i.e. they undergo change in volume whenever the pressure changes.

  The term **hydrodynamics** is often applied to the science of moving incompressible fluids. However, there is no sharp distinctions between the three states of matter i.e. solid, liquid and gases.

- In **microscopic view of fluids, matter is assumed to be composed of molecules** which are in random relative motion under the action of intermolecular forces.
- In solids, spacing of the molecules is small, spacing persists even under strong molecular forces.
- In liquids, the spacing between molecules is greater even under weaker molecular forces
- and in gases, the gaps are even larger.

- **In macroscopic** viewpoint in which we would not see the gaps between the molecules and the matter would appear to be continuously distributed.
- We shall take this macroscopic view of fluids in which physical quantities associated with the fluids within a given volume V are assumed to be distributed continuously and, within a sufficiently small volume δV, uniformly.
- This observation is known as **Continuum hypothesis.**
- It implies that at each point of a fluid, we can prescribe a unique velocity, a unique pressure, a unique density etc. Moreover, for a continuous or ideal fluid we can define a **fluid particle** as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

- **Stresses :** Two types of forces act on a fluid element. One of them is body force and other is surface force. The body force is proportional to the mass of the body on which it acts while the surface force is proportional to the surface area and acts on the boundary of the body.
- Suppose F is the surface force acting on an elementary surface area dS at a point P of the surface S. Let F1 and F2 be resolved parts of F in the directions of tangent and normal at P. The normal force per unit area is called the **normal stress** and is also called **pressure**. The tangential force per unit area is called the **shearing stress**.

- **Viscosity :** It is the internal friction between the particles of the fluid which offers resistance to the deformation of the fluid. The friction is in the form of tangential and shearing forces (stresses). Fluids with such property are called **viscous or real fluids** and those not having this property are called **inviscid or ideal or perfect fluids.**
- Actually, all fluids are real, but in many cases, when the rates of variation of fluid velocity with distances are small, viscous effects may be ignored.

- From the definition of body force and shearing stress, it is clear that body force per unit area at every point of surface of an ideal fluid acts along the normal to the surface at that point. Thus ideal fluid does not exert any shearing stress.
- Thus, we conclude that viscosity of a fluid is that property by virtue of which it is able to offer resistance to shearing stress. It is a kind of molecular frictional resistance.

- **Velocity of Fluid at a Point :** Suppose that at time t, a fluid particle is at the point P having position vector $\vec{r}\ (\vec{r} = \overrightarrow{OP})$ and at time t + δt the same particle has reached at point Q having position vector $\vec{r} + \delta\vec{r}$. The particle velocity q at point P is

$$\vec{q} = \lim_{\delta t \to 0} \frac{\overrightarrow{(r + \delta\vec{r})} - \vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

where the limit is assumed to exist uniquely. Clearly velocity is in general dependent on both position vector and time, so we may write $\vec{q} = \vec{q}(\vec{r}, t) = \vec{q}(x, y, z, t)$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (P has co-ordinates (x, y, z)).

- **Remarks.** (i) A point where $q = 0$, is called a stagnation point.
- (ii) When the flow is such that the velocity at each point is independent of time i.e. the flow pattern is same at each instant, then the motion is termed as steady motion, otherwise it is unsteady.

- **Flux across any surface :** The flux i.e. the rate of flow across any surface S is defined by the integral

$$\iint \rho(\vec{q}.\hat{n})dS$$

where ρ is the density, q is the velocity of the fluid and nˆ is the outward unit normal at any point of S.

Also, we define

Flux = density * normal velocity * area of the surface.

Eulerian and Lagrangian Methods
(Local and Total range of change)

- We have two methods for studying the general problem of fluid dynamics.
- **Eulerian Method :** In this method, we fix a point in the space occupied by the fluid and observation is made of whatever changes of velocity, density pressure etc take place at that point. i.e. point is fixed and fluid particles are allowed to pass through it. If P(x, y, z) is the point under reference, then x, y, z do not depend upon the time parameter t, therefore $\dot{x}, \dot{y}, \dot{z}$ do not exist (dot denotes derivative w.r.t. time t).

- Let f(x, y, z, t) be a scalar function associated with some property of the fluid (e.g. its density) i.e. f(x, y, z, t) = $f(\vec{r}, t)$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

  is the position vector of the point P, then

$$\frac{\partial f}{\partial t} = \lim_{\delta t \to 0} \frac{\overline{f(\vec{r} + \delta\vec{r}) - f(\vec{r})}}{\delta t} \qquad (1)$$

Here, $\frac{\partial f}{\partial t}$ is called local time rate of change.

Lagrangian Method

- In this case, observations are made at each point and each instant, i.e., any particle of the fluid is selected and observation is made of its particular motion and it is pursued throughout its course.
- Let a fluid particle be initially at the point (a, b, c). After lapse of time t, let the same fluid particle be at (x, y, z). It is obvious that x, y, z are functions of t.
- But since the particles which have initially different positions occupy different positions after the motion is allowed. Hence the co-ordinates of the final position i.e. (x, y, z) depend on (a, b c) also. Thus

  $x = f_1(a, b, c, t)$, $y = f_2(a, b, c, t)$, $z = f_3(a, b, c, t)$.

- For this case, if f(x, y, z, t) be scalar function associated with the fluid, then
- $$\frac{df}{dt} = \lim_{\delta t \to 0} \frac{f(\vec{r} + \delta\vec{r}, t + \delta t) - f(\vec{r}, t)}{\delta t}$$

where $\dot{x}, \dot{y}, \dot{z}$ exist.

$\frac{df}{dt}$ is called an individual time rate or total rate or particle rate of change.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right)\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt} = \nabla f . \vec{q} + \frac{\partial f}{\partial t}$$

Where,

$$\vec{q} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) = (u, v, w)$$

- Remarks. (i) The relation

$$\frac{df}{dt} = \nabla f . \vec{q} + \frac{\partial f}{\partial t}$$

implies $$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q} . \nabla$$

- The operator $\frac{d}{dt}$ is called Lagrangian operator or material derivative i.e. time rate of change in Lagrangian view. Sometimes, it is called 'differentiation following the fluid'.
- (ii) Similarly, for a vector function $\vec{F}(x, y, z, t)$ associated with some property of the fluid (e.g. its velocity, acceleration), we can show that

$$\frac{d\vec{F}}{dt} = \nabla\vec{F}.\vec{q} + \frac{\partial}{\partial t}\vec{F}$$

- Hence the relation holds for both scalar and vector functions associated with the moving fluid.

(iii) The Eulerian method is sometimes also called the flux method.

(iv) Both Lagrangian and Eulerian methods were used by Euler for studying fluid dynamics.

(v) Lagrangian method resembles very much with the dynamics of a particle

(vi) The two methods are essentially equivalent, but depending upon the problem, one has to judge whether Lagrangian method is more useful Sor the Eulerian.

Streamlines, Pathlines and Streaklines

- **Streamlines** : It is a curve drawn in the fluid such that the direction of the tangent to it at any point coincides with the direction of the fluid velocity vector $\vec{q}$ at that point. At any time t, let $\vec{q}$ = (u, v, w) be the velocity at each point P(x, y, z) of the fluid. The direction ratios of the tangent to the curve at P(x, y, z) are $d\vec{r} = (dx, dy, dz)$ since the tangent and the velocity at P have the same direction, therefore $\vec{q} \times d\vec{r} = 0.$
- $i.e.\ \left(u\hat{i} + v\hat{j} + w\hat{k}\right) \times \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$ =0.

$$(v\,dz - w\,dy)\hat{i} + (w\,dx - u\,dz)\hat{j} + (u\,dy - v\,dx)\hat{k} = \vec{0}.$$

So we get $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

- These are the differential equations for the streamlines. i.e. their solution gives the streamlines.

- **Pathlines:** When the fluid motion is steady so that the pattern of flow does not vary with time, the paths of the fluid particles coincide with the streamlines. But in case of unsteady motion, the flow pattern varies with time and the paths of the particles do not coincide with the streamlines. However, the streamline through any point P does touch the pathline through P. Pathlines are the curves described by the fluid particles during their motion i.e. these are the paths of the particles. The differential equations for pathlines are

$$\frac{d\vec{r}}{dt} = \vec{q} \text{ or } \frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w. \qquad (1)$$

- where now (x, y, z) are the Cartesian co-ordinates of the fluid particle and not a fixed point of space. The equation of the pathline which passes through the point (x_0, y_0, z_0), which is fixed in space, at time t = 0 say, is the solution of (1) which satisfy the initial condition that $x = x_0$, $y = y_0$, $z = z_0$ when $t = 0$. The solution gives a set of equations of the form

$$x = x(x_0, y_0, z_0, t)$$
$$y = y(x_0, y_0, z_0, t) \qquad (2)$$
$$z = z(x_0, y_0, z_0, t)$$

which, as t takes all values greater than zero, will trace out the required pathline.

- **Remarks** : (i) Streamlines give the motion of each particle at a given instant whereas pathlines give the motion of a given particle at each instant. We can make these observations by using a suspension of aluminum dust in the liquid.
- (ii) If we draw the streamlines through every point of a closed curve in the fluid, we obtain a stream tube. A stream tube of very small crosssection is called a stream filament.

(iii) The components of velocity at right angle to the streamline is always zero. This shows that there is no flow across the streamlines. Thus, if we replace the boundary of stream tube by a rigid boundary, the flow is not affected. The principle of conservation of mass then gives that the flux across any cross-section of the stream tube should be the same.

- **Streaklines** : In addition to streamlines and pathlines, it is useful for observational purpose to define a streakline. This is the curve of all fluid particles which at some time have coincided with a particular fixed point of space. Thus, a streakline is the locus of different particles passing through a fixed point. The streakline is observed when a neutrally buoyant marker fluid is continuously injected into the flow at a fixed point of space from time $t = -\infty$. The marker fluid may be smoke if the main flow involves a gas such as air, or a dye such as potassium permanganate ($KMnO_4$) if the main flow involves a liquid such as water.

- If the co-ordinates of a particle of marker fluid are (x, y, z) at time t and the particle coincided with the injection point (x_0, y_0, z_0) at some time τ, where $\tau \le t$, then the time-history (streakline) of this particle is obtained by solving the equations for a pathline, subject to the initial condition that $x = x_0$, $y = y_0$, $z = z_0$ at $t = \tau$. As t takes all possible values in the angle $-\infty \le \tau \le t$, the locations of all fluid particles on the streakline through (x_0, y_0, z_0) are obtained. Thus, the equation of the streakline at time t is given by

$$x = x(x_0, y_0, z_0, t, \tau)$$

$$y = y(x_0, y_0, z_0, t, \tau) \qquad -\infty \leq \tau \leq t$$

$$z = z(x_0, y_0, z_0, t, \tau)$$

- Remark: (i) For a steady flow, streaklines also coincide with streamlines and pathlines.
- (ii) Streamlines, pathlines and streaklines are termed as flowlines for a fluid.

Velocity Potential

- Suppose that $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ is the velocity at any time t at each point P(x, y, z) of the fluid. Also suppose that the expression u dx + vdy + wdz is an exact differential, say - df.
- Then, -df = udx + vdy + wdz
- i.e. $-\left(\frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz + \frac{\partial\emptyset}{\partial t}dt\right) = udx + vdy + wdz, where\ \emptyset = \emptyset(x,y,z,t)$ is some scalar function, uniform throughout the entire field of flow.

$$u = -\frac{\partial \emptyset}{\partial x}, v = -\frac{\partial \emptyset}{\partial y}, w = -\frac{\partial \emptyset}{\partial z}, \quad \frac{\partial \emptyset}{\partial t} = 0 \Rightarrow \emptyset = \emptyset(x,y,z)$$

$$\vec{q} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} = -\left(\frac{\partial \emptyset}{\partial x}\hat{\imath} + \frac{\partial \emptyset}{\partial y}\hat{\jmath} + \frac{\partial \emptyset}{\partial z}\hat{k}\right) = -\nabla \emptyset$$

Φ, is termed as the velocity potential and the flow of such type is called flow of potential kind.

- In the above definition, the negative sign in $\vec{q} = -\nabla \emptyset$ is a convention and it ensures that flow takes place from higher to lower potentials. The level surfaces $\emptyset(x,y,z)$ = constant, are called **equipotentials or equipotential surfaces.**

- **Theorem :** At all points of the field of flow the equipotentials (i.e. equipotential surfaces) are cut orthogonally by the streamlines.
- **Proof.** If the fluid velocity at any time t be q = (u, v, w), then the equations of streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \qquad (1)$$

The surfaces given $\vec{q} \cdot d\vec{r} = 0$ i.e. $u dx + v dy + w dz = 0.$ (2)

are such that the velocity is at right angles to the tangent planes. The curves (1) and the surfaces (2) cut each other orthogonally. Suppose that the expression on the left hand side of (2) is an exact differential, say, -d Φ, then

$$d\phi = udx + vdy + wdz$$

- where Φ is velocity potential. The necessary and sufficient condition for the relations.
- $u = -\frac{\partial \phi}{\partial x}, v = -\frac{\partial \phi}{\partial y}, w = -\frac{\partial \phi}{\partial z}$ or $\vec{q} = -\nabla \phi$ (3)
- $curl\,\vec{q} = curl(-\nabla \phi) = \overline{0}.$ (4)

The solution of (2) i.e. dΦ = 0 is

$$\Phi(x, y, z) = \text{const.} \qquad (5)$$

- The surfaces (5) are called equipotentials. Thus the equipotentials are cut orthogonally by the stream lines.