DIFFERENTIAL EQUATIONS

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Differential Equations

- An equation which involves unknown function and its derivatives
  - ordinary differential equation (ode) : not involve partial derivatives
  - partial differential equation (pde) : involves partial derivatives
  - ***order*** of the differential equation is the order of the highest derivatives

Examples:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x\sin y$$

→ second order ordinary differential equation

$$\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial x} = \frac{x+t}{x-t}$$

→ first order partial differential equation

Modeling via Differential Equations

Note that the set of equations is called a **Model for the system.**

How do we build a Model?

The basic steps in building a model are:

Step 1: Clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied.

Step 2: Completely describe the parameters and variables to be used in the model.

Step 3: Use the assumptions (from Step 1) to derive mathematical equations relating the parameters and variables (from Step 2).

Differential Equations

- ⇨ A linear differential equation of order n is a differential equation written in the following form:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)$$

where $a_n(x)$ is not the zero function

- ⇨ **General solution** : looking for the unknown function of a differential equation
- ⇨ **Particular solution (Initial Value Problem)** : looking for the unknown function of a differential equation where the values of the unknown function and its derivatives at some point are known
- ⇨ Issues in finding solution : **existence** and **uniqueness**

1^{st} Order DE - Separable Equations

The differential equation *M(x,y)dx* + *N(x,y)dy* = 0 is separable if the equation can be written in the form:

$$f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$$

Solution :

1. Multiply the equation by integrating factor: $\dfrac{1}{f_2(x)g_1(y)}$
2. The variable are separated :

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0$$

3. Integrating to find the solution:

$$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy = C$$

1^{st} Order DE - Separable Equations

Examples:

1. Solve : $4x\,dy - y\,dx = x^2 dy$

   Answer:

1^{st} Order DE - Separable Equations

Examples:

1. Solve : $$\frac{dy}{dx} = \frac{x^2 + 2}{y}$$

   Answer:

1st Order DE - Separable Equations

Examples:

2. Find the particular solution of : $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x}$; $y(1) = 2$

Answer:

1^{st} Order DE - Homogeneous Equations

Homogeneous Function

f (x,y) is called homogenous of degree *n* if : $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

Examples:

$f(x,y) = x^4 - x^3 y$ → homogeneous of degree 4

$$f(\lambda x, \lambda y) = (\lambda x)^4 - (\lambda x)^3 (\lambda y)$$
$$= \lambda^4 (x^4 - x^3 y) = \lambda^4 f(x,y)$$

$f(x,y) = x^2 + \sin x \cos y$ → non-homogeneous

$$f(\lambda x, \lambda y) = (\lambda x)^2 + \sin(\lambda x)\cos(\lambda y)$$
$$= \lambda^2 x^2 + \sin(\lambda x)\cos(\lambda y)$$
$$\neq \lambda^n f(x,y)$$

1^{st} Order DE - Homogeneous Equations

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree

Solution :

1. Use the transformation to : $y = vx \quad \Rightarrow \quad dy = v\,dx + x\,dv$
2. The equation become separable equation:

$$P(x,v)dx + Q(x,v)dv = 0$$

3. Use solution method for separable equation

$$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(v)}{g_1(v)}dv = C$$

4. After integrating, v is replaced by y/x

1st Order DE – Homogeneous Equations

Examples:

1. Solve : $\left(x^3 + y^3\right)dx - 3xy^2\ dy = 0$

   Answer:

1st Order DE - Homogeneous Equations

Examples:

2. Solve : $$\frac{dy}{dx} = \frac{-2x + 5y}{2x + y}$$

Answer:

1st Order DE – Exact Equation

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if : $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

The solutions are given by the implicit equation $F(x,y) = C$
where : $\partial F/\partial x = M(x,y)$ and $\partial F/\partial y = N(x,y)$

Solution :

1. Integrate either $M(x,y)$ with respect to x or $N(x,y)$ to y. Assume integrating $M(x,y)$, then :

$$F(x,y) = \int M(x,y)dx + \theta(y)$$

2. Now : $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)dx\right] + \theta'(y) = N(x,y)$$

or : $$\theta'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M(x,y)dx\right]$$

1^{st} Order DE – Exact Equation

3. Integrate $\theta'(y)$ to get $\theta(y)$ and write down the result $F(x,y) = C$

Examples:

1. Solve : $\left(2x^3 + 3y\right)dx + (3x + y - 1)dy = 0$

   Answer:

1st Order DE – Exact Equation

Examples:

2. Solve : $$4xy + 1 + \left(2x^2 + \cos y\right)\frac{dy}{dx} = 0$$

Answer:

1st Order DE – Non Exact Equation

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is a non exact equation if : $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

The solutions are given by using integrating factor to change the equation into exact equation

Solution :

1. Check if : $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \Rightarrow \text{function of } x \text{ only}$$

then integrating factor is $e^{\int f(x)dx}$

or if : $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \Rightarrow \text{function of } y \text{ only}$$

then integrating factor is $e^{\int g(y)dy}$

1st Order DE – Non Exact Equation

2. Multiply the differential equation with integrating factor which result an exact differential equation
3. Solve the equation using procedure for an exact equation

1^{st} Order DE – Non Exact Equation

Examples:

1. Solve : $\left(x^2 + y^2 + x\right)dx + xy\, dy = 0$

   Answer:

1^{st} Order DE – Non Exact Equation

Examples:

2. Solve : $$\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}$$

Answer:

1^{st} Order DE – Linear Equation

A **first order linear differential equation** has the following general form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Solution :

1. Find the integrating factor: $u(x) = e^{\int p(x)dx}$
2. Evaluate : $\int u(x)q(x)dx$
3. Find the solution:

$$y = \frac{\int u(x)q(x)dx + C}{u(x)}$$

1^{st} Order DE – Linear Equation

Examples:

1. Solve : $\dfrac{dy}{dx} + 2xy = 4x$

   Answer:

1st Order DE – Linear Equation

Examples:

2. Find the particular solution of :

$$y' + \tan(x)y = \cos^2(x), \quad y(0) = 2$$

Answer:

2^{nd} Order DE – Linear Equation

A **second order** differential equation is an equation involving the unknown function *y*, its derivatives *y*' and *y*'', and the variable *x*. We will consider explicit differential equations of the form :

$$\frac{d^2y}{dx^2} = f(y, y', x)$$

A **linear** second order differential equations is written as:

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

When $d(x) = 0$, the equation is called **homogeneous**, otherwise it is called **nonhomogeneous**

2^{nd} Order DE – Linear Equation

To a nonhomogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = d(x) \qquad (NH)$$

we associate the so called **associated homogeneous equation**

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (H)$$

Main result:

The general solution to the equation (NH) is given by:

$$y = y_h + y_p$$

where:

(i) y_h is the general solution to the associated homogeneous equation (*H*);

(ii) y_p is a particular solution to the equation (*NH*).

2^{nd} Order DE – Linear Equation

Basic property

Consider the homogeneous second order linear equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

or the explicit one

$$y'' + p(x)y' + q(x)y = 0$$

Property:

If y_1 and y_2 are two solutions, then:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution for any arbitrary constants c_1 ,c_2

2nd Order DE – Reduction of Order

Reduction of Order Technique

This technique is very important since it helps one to find **a second solution** independent from **a known one**.

Let y_1 be a non-zero solution of: $y'' + p(x)y' + q(x)y = 0$

Then, a second solution y_2 independent of y_1 can be found as:

$$y_2(x) = y_1(x)v(x)$$

Where:

$$v(x) = \int \frac{1}{y_1^2(x)} e^{-\int p(x)dx} dx$$

The general solution is then given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

2^{nd} Order DE – Reduction of Order

Examples:

1. Find the general solution to the Legendre equation

$$\left(1-x^2\right)y'' - 2xy' + 2y = 0$$

Using the fact that : $y_1 = x$ is a solution.

2^{nd} Order DE – Homogeneous LE with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients

A second order homogeneous equation with constant coefficients is written as: $ay'' + by' + cy = 0 \qquad (a \neq 0)$

where *a*, *b* and *c* are constant

The steps to follow in order to find the general solution is as follows:

(1) *Write down the* ***characteristic equation***

$$a\lambda^2 + b\lambda + c = 0 \qquad (a \neq 0)$$

This is a quadratic. Let λ_1 and λ_2 be its roots we have

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2^{nd} Order DE – Homogeneous LE with Constant Coefficients

(2) *If λ_1 and λ_2 are distinct real numbers (if **b2 - 4ac > 0**), then the general solution is:*

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

(3) *If λ_1 = λ_2 (if **b2 - 4ac = 0**), then the general solution is:*

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

(4) *If λ_1 and λ_2 are complex numbers (if **b2 - 4ac < 0**), then the general solution is:*

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

Where:

$$\alpha = \frac{-b}{2a} \text{ and } \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

2^{nd} Order DE – Homogeneous LE with Constant Coefficients

1. Find the solution to the Initial Value Problem

$$y'' + 2y' + 2y = 0 \;\; ; \;\; y(\pi/4) = 2 \;\; ; \;\; y'(\pi/4) = -2$$

2^{nd} Order DE – Non Homogeneous LE

To a nonhomogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = d(x) \qquad (NH)$$

we associate the so called **associated homogeneous equation**

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (H)$$

Main result:

The general solution to the equation (NH) is given by:

$$y = y_h + y_p$$

where:

(i) y_h is the general solution to the associated homogeneous equation (*H*);

(ii) y_p is a particular solution to the equation (*NH*).

2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

We will guess the form of y_p and then plug it in the equation to find it. However, it works only under the following two conditions:

- ❑ the associated homogeneous equations has constant coefficients
- ❑ the nonhomogeneous term $d(x)$ is a special form

$$d(x) = P_n(x)e^{\alpha x}\cos(\beta x) \quad \text{or} \quad d(x) = L_n(x)e^{\alpha x}\sin(\beta x)$$

where $P_n(x)$ and $L_n(x)$ are polynomial functions of degree n

Note: we may assume that $d(x)$ is a sum of such functions

Then a particular solution yp is given by:

$$y_p(x) = x^s\left(T_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)\right)$$

Where:

$$T_n(x) = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n$$
$$R_n(x) = B_0 + B_1x + B_2x^2 + \ldots + B_nx^n$$

2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

The steps to follow in applying this method:

1. Check that the two conditions are satisfied
2. Write down the characteristic equation and find its root
$$a\lambda^2 + b\lambda + c = 0$$
3. Write down the number $\alpha + \beta i$
4. Compare this number to the roots of the characteristic equation

$$\text{If :}\quad \begin{aligned} &\alpha + \beta i \text{ is not one of the roots} && \Rightarrow s = 0 \\ &\alpha + \beta i \text{ is one of the distinc roots} && \Rightarrow s = 1 \\ &\alpha + \beta i \text{ is equal to both roots} && \Rightarrow s = 2 \end{aligned}$$

5. Write down the form of particular solution
$$y_p(x) = x^s\left(T_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)\right)$$
Where: $T_n(x) = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n \quad R_n(x) = B_0 + B_1x + B_2x^2 + \ldots + B_nx^n$
6. Find constant *A* and *B* by plugging y_p solution to original equation

2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

1. Find a particular solution to the equation

$$y'' - 3y' - 4y = 2\sin(x)$$

2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

If the nonhomogeneous term $d(x)$ consist of several terms:

$$d(x) = d_1(x) + d_2(x) + \ldots + d_N(x) = \sum_{i=1}^{i=N} d_i(x)$$

We split the original equation into N equations

$$a(x)y'' + b(x)y' + c(x)y = d_1(x)$$
$$a(x)y'' + b(x)y' + c(x)y = d_2(x)$$
$$\vdots$$
$$a(x)y'' + b(x)y' + c(x)y = d_N(x)$$

Then find a particular solution y_{pi}

A particular solution to the original equation is given by:

$$y_p(x) = y_{p1}(x) + y_{p2}(x) + \ldots + y_{pN}(x) = \sum_{i=1}^{i=N} y_{pi}(x)$$

2^{nd} Order DE – Non Homogeneous LE Method Undetermined Coefficients

If the nonhomogeneous term $d(x)$ consist of several terms:

$$d(x) = d_1(x) + d_2(x) + \ldots + d_N(x) = \sum_{i=1}^{i=N} d_i(x)$$

We split the original equation into N equations

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$$a(x)y'' + b(x)y' + c(x)y = d_2(x)$$
$$\vdots$$
$$a(x)y'' + b(x)y' + c(x)y = d_N(x)$$

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