Calculus of Variations

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Calculus of Variations

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V-1 Maximum and Minimum of Functions

Part A Functional ⟶ Euler's equation

Maximum and Minimum of functions

(a) If f(x) is twice continuously differentiable on $[x_0 , x_1]$ i.e.

Nec. Condition for a max. (min.) of f(x) at $x \in [x_0, x_1]$ is that $F'(x) = 0$

Suff. Condition for a max (min.) of f(x) at $x \in [x_0, x_1]$ are that $F'(x) = 0$ also $F''(x) \le 0$ ($F''(x) \ge 0$)

(b) If f(x) over closed domain D. Then nec. and suff. Condition for a max. (min.)

of f(x) at $x_0 \in D - \partial D$ are that $\frac{\partial f}{\partial x_i}\Big|_{x=x_0} = 0$ i = 1,2…n and also that $\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_{x=x_0}$ is a negative infinite .

(c) If f(x) on closed domain D

If we want to extremize f(x) subject to the constraints

$g_i(x_1,\ldots,x_n)=0$ i=1,2,…k (k < n)

Ex :Find the extrema of f(x,y) subject to g(x,y) =0

(i) 1st method : by direct diff. of g

$$dg = g_x dx + g_y dy = 0$$

$$\Rightarrow \quad dy = -\frac{g_x}{g_y} dx$$

To extremize f

$$df = f_x dx + f_y dy = 0$$

$$\Rightarrow \quad (f_x - f_y \frac{g_x}{g_y}) dx = 0$$

We have

$$f_x g_y - f_y g_x = 0 \quad \text{and} \quad g = 0$$

to find (x_0,y_0) which is to extremize f subject to g = 0

(ii) 2nd method : (Lagrange Multiplier)

Let $$v(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$\Rightarrow$ extrema of v without any constraint

$\Leftrightarrow$ extrema of f subject to g = 0

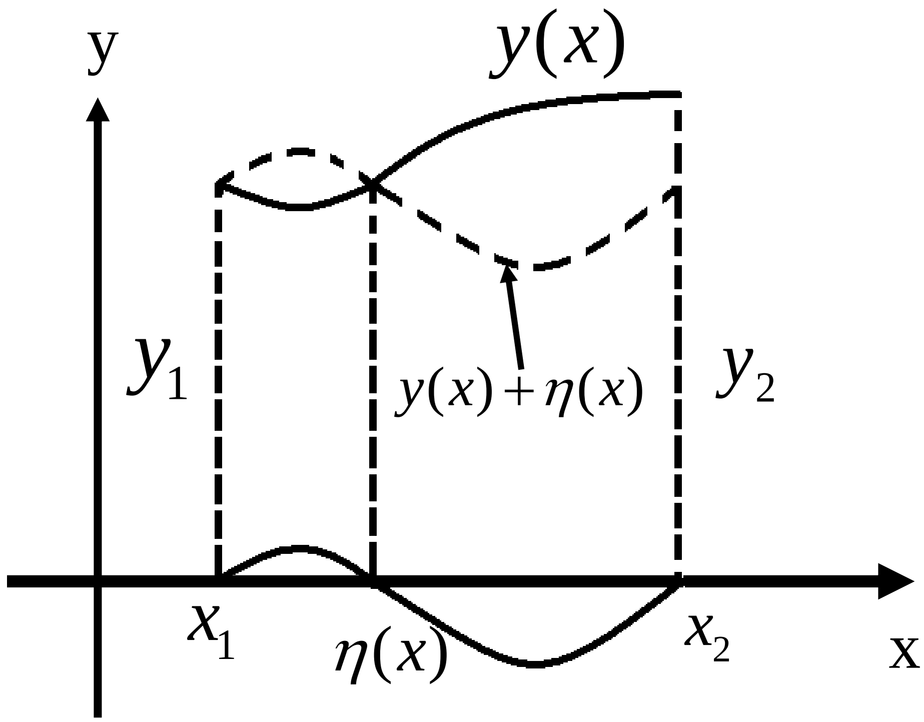
To extremize v $\Rightarrow$
$$\begin{cases} \dfrac{\partial v}{\partial x} = f_x + \lambda g_x = 0 \\ \dfrac{\partial v}{\partial y} = f_y + \lambda g_y = 0 \end{cases} \Rightarrow \begin{array}{l} f_x g_y - f_y g_x = 0 \\ \dfrac{\partial v}{\partial \lambda} = g = 0 \end{array}$$

We obtain the same equations to extrimizing. Where λ is called The Lagrange Multiplier.

V-2 Maximum and Minimum of Functionals

2.1 What are functionals.

Functional is function's function.



2.2 The simplest problem in calculus of variations.

Determine $y(x) \in c^2[x_0, x_1]$ such that the functional :

$$I\left(y(x)\right) = \int_{x_0}^{x_1} F\left(x, y(x), \dot{y}(x)\right) dx \quad \text{as an extrema}$$

where $F \in c^2$ over its entire domain , subject to $y(x_0) = y_0$, $y(x_1) = y_1$ at the end points.

On integrating by parts of the 2nd term

$$\Rightarrow [F_{y'}(x,y,y')\eta]_{x_0}^{x_1} - \int_{x_0}^{x_1} [\frac{d}{dx} F_{y'}(x,y,y') - F_y(x,y,y')]\eta dx = 0 \quad \text{------- (1)}$$

since $\eta(x_0) = \eta(x_1) = 0$ and since $\eta(x)$ is arbitrary.

$$\Rightarrow \frac{d}{dx}\left[F_{y'}(x,y,y')\right] - F_y(x,y,y') = 0 \quad \text{-------- (2)} \quad \text{Euler's Equation}$$

Natural B.C's

$$\left[\frac{\partial F}{\partial y'}\right]_{x_0} = 0 \quad \text{or /and} \quad \left[\frac{\partial F}{\partial y'}\right]_{x=x_1} = 0$$

The above requirements are called national b.c's.

V-3 The Variational Notation

Variations

Imbed u(x) in a "parameter family" of function $\phi(x,\varepsilon)=u(x)+\varepsilon\eta(x)$ the variation of u(x) is defined as

$$\delta u=\varepsilon\eta(x)$$

The corresponding variation of F , δF to the order in ε is ,

since $$\delta F=F(x+y+\varepsilon\eta,y'+\varepsilon\eta')-F(x,y,y')$$

$$=\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'$$

and $$I(u+\varepsilon\eta)=\int_{x_0}^{x_1}F(x,u+\varepsilon\eta,u'+\varepsilon\eta')dx=G(\varepsilon)$$

Then $$\delta I=\delta\int_{x_0}^{x_1}F(x,y,y')dx$$

$$=\int_{x_0}^{x_1}\delta F(x,y,y')dx$$

$$=\int_{x_0}^{x_1}\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right)dx$$

$$= \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \delta y dx + \left[\frac{\partial F}{\partial y'} \delta y \right]_{x_0}^{x_1}$$

Thus a stationary function for a functional is one for which the first variation = 0.

$\frac{\partial F}{\partial y}$

<u>For the more general cases</u>

(a) Several dependent variables.

Ex : $I = \int_{x_0}^{x_1} F(x, y, z\, ; y', z')dx$

Euler's Eq. $\Longrightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0 \quad , \quad \frac{\partial F}{\partial z} - \frac{d}{dx}(\frac{\partial F}{\partial z'}) = 0$

(b) Several Independent variables.

Ex : $I = \iint_R F(x, y, u, u_x, u_y)dxdy$

Euler's Eq. $\Longrightarrow \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y}) = 0$

(C) High Order.

Ex : $$I = \int_{x_0}^{x_1} F(x, y, y', y'')dx$$

Euler's Eq. $$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

- Variables ↑ → Causing more equation.
- Order ↑ → Causing longer equation.

V-4 Constraints and Lagrange Multiplier

Lagrange multiplier

Lagrange multiplier can be used to find the extreme value of a multivariate function f subjected to the constraints.

Ex :

(a) Find the extreme value of $I = \int_{x_0}^{x_1} F(x, u, v, u_x, v_x) dx$

where $u(x_1) = u_1 \qquad u(x_2) = u_2$

$v(x_1) = v_1 \qquad v(x_2) = v_2$

and subject to the constraints

$$G(x, u, v) = 0 \qquad \text{------------}(1)$$

From $\delta I = \int_{x_1}^{x_2} \left\{ \left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x} \right) \right] \delta u + \left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x} \right) \right] \partial v \right\} dx = 0$ -----------(2)

Because of the constraints, we don't get two Euler's equations.

From

$$\delta G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial v}\delta v = 0$$

$$\Rightarrow \quad -\frac{G_v}{G_u}\delta v = \delta u$$

So (2)

$$\Rightarrow \quad \delta I = \int_{x_0}^{x_1}\left\{-\frac{G_v}{G_u}\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\right] + \left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\right]\right\}\delta v dx = 0$$

$$\Rightarrow \quad \frac{\partial G}{\partial v}\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\right] - \frac{\partial G}{\partial u}\left[\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right)\right] = 0$$

The above equations together with (1) are to be solved for u , v.

(b) Simple Isoparametric Problem

To extremize $I = \int_{x_1}^{x_2} F(x, y, y')dx$, subject to the constraint :

(1) $J = \int_{x_1}^{x_2} G(x, y, y')dx = const.$

(2) $y(x_1) = y_1$, $y(x_2) = y_2$

Take the variation of two-parameter family : $y + \delta y = y + \varepsilon_1 \eta_1(x) + \varepsilon_2 \eta_2(x)$
(where $\eta_1(x)$ and $\eta_2(x)$ are some equations which satisfy
$\eta_1(x_1) = \eta_2(x_1) = \eta_1(x_2) = \eta_2(x_2) = 0$)

Then , $I(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} F(x, y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, y' + \varepsilon_1\eta_1' + \varepsilon_2\eta_2')dx$

$J(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} G(x, y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, y' + \varepsilon_1\eta_1' + \varepsilon_2\eta_2')dx$

To base on Lagrange Multiplier Method we can get :

$$\frac{\partial}{\partial \varepsilon_1}\left(I+\lambda J\right)\Big|_{\varepsilon_1=\varepsilon_2=0}=0$$

$$\frac{\partial}{\partial \varepsilon_2}\left(I+\lambda J\right)\Big|_{\varepsilon_1=\varepsilon_2=0}=0$$

$$\Rightarrow \quad \int_{x_1}^{x_2}\left\{\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]+\lambda\left[\frac{\partial G}{\partial y}-\frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right]\right\}\eta_i dx=0 \qquad \text{i = 1,2}$$

So the Euler equation is：

$$\frac{\partial}{\partial y}\left(F+\lambda G\right)-\frac{d}{dx}\left[\frac{\partial}{\partial y'}\left(F+\lambda G\right)\right]=0$$

when $\frac{\partial G}{\partial y}-\frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)=0$, λ is arbitrary numbers.

$\Rightarrow$ The constraint is trivial, we can ignore λ .

Examples

Euler's equation ⟶ Functional

Helmholtz Equation

Ex : Force vibration of a membrane.

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y, t) \qquad \text{-----(1)}$$

if the forcing function f is of the form

$$f(x, y, t) = P(x, y)\sin(\omega t + \alpha)$$

we may write the steady state disp u in the form

$$u = v(x, y)\sin(\omega t + \alpha)$$

(1) $\Rightarrow$

$$c^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \omega^2 v + p = 0$$

$$\int_R \left[c^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \omega^2 v + p \right] \delta v \, dxdy = 0 \quad \text{-----(2)}$$

Consider

$$c^2 \int_R v_{xx} \delta v \, dxdy$$
$$= c^2 \int_R [(v_x \delta v)_x - v_x \delta v_x] dxdy$$

Note that $(v_x \delta v)_x = v_{xx} \delta v + v_x \delta v_x$

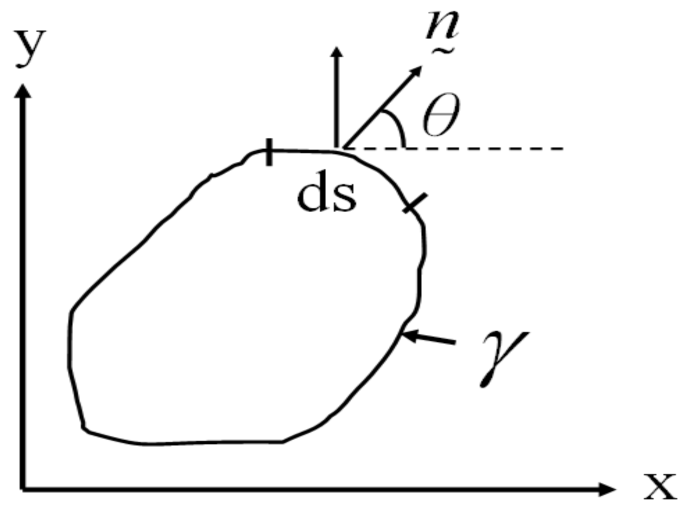
$$c^2 \int_R v_{yy} \delta v \, dxdy$$
$$= c^2 \int_R [(v_y \delta v)_y - v_y \delta v_y] dxdy$$

$$(v_y \delta v)_y = v_{yy} \delta v + v_y \delta v_y$$

$$V = V_x i + V_y j \qquad V_x = v_x \delta v \qquad V_y = v_y \delta v$$

$$\underset{\sim}{n} = \cos\theta\, i + \sin\theta\, j$$

$$\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial (v_x \delta v)}{\partial x} + \frac{\partial (v_y \delta v)}{\partial y}$$



$$\int_{\Re}(\nabla \cdot V)da = \oint_{\gamma} V \cdot n\, ds$$

$$= \oint (v_x \delta v \cos\theta + v_y \delta v \sin\theta)\mathrm{ds}$$

$$c^2\int_R v_{xx}\delta v dxdy + \int_R c^2 v_{yy}\delta v dxdy$$

$$= c^2\oint_{\gamma} v_x \delta v \cos\theta ds - \int_R \frac{1}{2}c^2\delta(v_x)^2 dxdy$$

$$+ c^2\oint_{\gamma} v_y \delta v \sin\theta ds - \int_R \frac{1}{2}c^2\delta(v_y)^2 dxdy$$

$$\Rightarrow \oint_{\gamma} c^2(v_x\cos\theta + v_y\sin\theta)\delta v ds - \int_R \frac{1}{2}c^2\delta[(v_x)^2 + (v_y)^2]dxdy$$

$$(2) \quad + \int_R \frac{1}{2}\omega_2\delta(v^2)dxdy + \int_R P\delta v dxdy = 0$$

$$\Rightarrow \int_{\gamma} c^2\frac{\partial v}{\partial n}\delta v ds - \delta\int_R [\frac{1}{2}c^2(\nabla v)^2 - \frac{1}{2}\omega^2 v^2 - Pv]dxdy = 0$$

Hence :

(i) if $v = f(x,y)$ is given on γ

i.e. $\delta v = 0$ on γ

then the variational problem

$$\Rightarrow \delta \int_R [\frac{c^2}{2}(\nabla v)^2 - \frac{1}{2}\omega^2 v^2 - pv] dxdy = 0 \qquad \text{-----(3)}$$

(ii) if $\frac{\partial v}{\partial n} = 0$ is given on γ

the variation problem is same as (3)

(iii) if $\frac{\partial v}{\partial n} = \psi(s)$ is given on γ

$$\Rightarrow \delta [\int_R \left\{ \frac{1}{2} c^2 (\nabla v)^2 - \frac{1}{2}\omega^2 v^2 - pv \right\} dxdy - \int_\tau c^2 \psi v dx] = 0$$

Diffusion Equation

Ex : Steady state Heat condition

$$\nabla \cdot (k \nabla T) = f(x, T) \qquad \text{in} \quad D$$

B.C's :

$$T = T_1 \qquad \text{on} \quad B_1$$
$$-kn \cdot \nabla T = q_2 \qquad \text{on} \quad B_2$$
$$-kn \cdot \nabla T = h(T - T_3) \qquad \text{on} \quad B_3$$

multiply the equation by δT , and integrate over the domain D. After integrating by parts, we find the variational problem as follow.

$$\delta\Big[\int_D \left\{ \frac{1}{2} k (\nabla T)^2 + \int_{T_0}^{T} f(x, T') dT' \right\} d\tau + \int_{B_2} q_2 T d\sigma + \frac{1}{2} \int_{B_3} h (T - T_3)^2 d\sigma \Big] = 0$$

with $T = T_1$ on B_1

Poison Equation

Ex : Torsion of a primatri Bar

$$\nabla^2 \psi = -2 \quad \text{in} \quad R$$
$$\psi = 0 \quad \text{on} \quad \gamma$$

where ψ is the Prandtl stress function and

$$\sigma_{\tau z} = G_\alpha \frac{\partial \psi}{\partial y} \quad , \quad \sigma_{zy} = \sigma_\alpha \frac{\partial \psi}{\partial x}$$

The variation problem becomes

$$\delta \left\{ \int_R [(D\psi)^2 - 4\psi] dxdy \right\} = 0$$

with $\psi = 0$ on γ

(I) Method of Weighted Residuals (MWR)

$$L[u] = 0 \quad \text{in} \quad D$$

+homo. b.c's in B

Assume approx. solution

$$u = u_n = \sum_{i=1}^{n} C_i \phi_i$$

where each trial function ϕ_i satisfies the b.c's

The residual

$$R_n = L[u_n]$$

In this

method (MWR), C_i are chosen such that Rn is forced to be zero in an average sense.

i.e. $< w_j, R_n > = 0$, $j = 1,2,\ldots,n$

where w_j are the weighting functions..

(II) Galerkin Method

w_j are chosen to be the trial functions ϕ_j hence the trial functions is chosen as members of a complete set of functions.

Galerkin method force the residual to be zero w.r.t. an orthogonal complete set.

Ex: Torsion of a square shaft

$$\nabla^2\psi = -2$$

$$\psi = 0 \quad on \quad x = \pm a, \quad y = \pm a$$

(i) one – term approximation

$$\psi_1 = c_1(x^2 - a^2)(y^2 - a^2)$$

$$R_i = \nabla^2\psi_1 + 2 = 2c_1[(x-a)^2 + (y-a)^2] + 2$$

$$\phi_1 = (x^2 - a^2)(y^2 - a^2)$$

From $\int_{-a}^{a}\int_{-a}^{a} R_1\phi_1 dxdy = 0$

$$\Rightarrow \quad c_1 = \frac{5}{8}\frac{1}{a^2}$$

therefore

$$\psi_1 = \frac{5}{8a^2}(x^2 - a^2)(y^2 - a^2)$$

the torsional rigidity

$$D_1 = 2G\int_R \psi dxdy = 0.1388G(2a)^4$$

the exact value of D is

$$D_a = 0.1406G(2a)^4$$

the error is only -1.2%

(ii) two – term approximation

$$\psi_2 = (x^2 - a^2)(y^2 - a^2)[c_1 + c_2(x^2 + y^2)]$$

⇩

By symmetry → even functions

$$\Rightarrow \quad R_2 = \nabla \psi_2 + 2$$

$$\phi_1 = (x^2 - a^2)(y^2 - a^2)$$

$$\phi_2 = (x^2 - a^2)(y^2 - a^2)(x^2 + y^2)$$

From $\int_R R_2 \phi_1 dxdy = 0$

and $\int_R R_2 \phi_2 dxdy = 0$

we obtain

$$c_1 = \frac{1295}{2216}\frac{1}{a^2}, \quad c_2 = \frac{525}{4432}\frac{1}{a^2}$$

therefore

$$D_2 = 2G\int_R \psi_2 dxdy = 0.1404G(2a)^4$$

the error is only -0.14%

(I) Kantorovich Method

Assuming the approximate solution as : $u = \sum_{i=1}^{n} C_i(x_n) U_i$

where U_i is a known function decided by b.c. condition.

C_i is a unknown function decided by minimal "I". $\Longrightarrow$ Euler Equation of C_i

Ex : The torsional problem with a functional "I".

$$I(u) = \int_{-a}^{a} \int_{-b}^{b} [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 - 4u] dxdy$$

Assuming the one-term approximate solution as :

$$u(x,y) = (b^2 - y^2)C(x)$$

Then,

$$I(\mathrm{C}) = \int_{-a}^{a}\int_{-b}^{b}\{(b^2 - y^2)^2[\mathrm{C}'(x)]^2 + 4y^2\mathrm{C}^2(x) - 4(b^2 - y^2)\mathrm{C}(x)\}dxdy$$

Integrate by y

$$I(\mathrm{C}) = \int_{-a}^{a}[\frac{16}{15}b^5\mathrm{C}'^2 + \frac{8}{3}b^3\mathrm{C}^2 - \frac{16}{3}b^3\mathrm{C}]dx$$

Euler's equation is

$$\mathrm{C}''(x) - \frac{5}{2b^2}\mathrm{C}(x) = -\frac{5}{2b^2} \quad \text{where b.c. condition is} \quad \mathrm{C}(\pm a) = 0$$

General solution is

$$\mathrm{C}(x) = \mathrm{A}_1\cosh(\sqrt{\frac{5}{2}}\frac{x}{b}) + \mathrm{A}_2\sinh(\sqrt{\frac{5}{2}}\frac{x}{b}) + 1$$

where $$A_1 = -\frac{1}{\cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)}, \quad A_2 = 0$$

and

$$C(x) = \left\{1 - \frac{\cosh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right)}{\cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)}\right\}$$

So, the one-term approximate solution is

$$u = \left\{1 - \frac{\cosh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right)}{\cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)}\right\}(b^2 - y^2)$$

(II) Raleigh-Ritz Method

This is used when the exact solution is impossible or difficult to obtain.

First, we assume the approximate solution as : $u = \sum_{i=1}^{n} C_i U_i$

Where, U_i are some approximate function which satisfy the b.c's.
Then, we can calculate extreme I .

$I = I(c_1,\ldots,c_n)$ choose $c_1 \sim c_n$ i.e. $\frac{\partial I}{\partial c_1} = \ldots = \frac{\partial I}{\partial c_n} = 0$

Ex: $y'' + xy = -x$, $y(0) = y(1) = 0$

Sol :

From

$$\int_0^1 (y'' + xy + x)\delta y dx = 0 \Rightarrow I = \int_0^1 \left[\frac{1}{2}(y')^2 - \frac{1}{2}xy^2 - xy \right] dx$$

Assuming that

$$y = x\left(1 - x\right)\left(c_1 + c_2 x + c_3 x^2 \ldots\right)$$

(1)One-term approx

$$y = c_1 x\left(1 - x\right) = c_1\left(x - x^2\right) \qquad y' = c_1\left(1 - 2x\right)$$

Then, $I\left(c_1\right) = \int_0^1 \left[\frac{1}{2} c_1^{\,2}\left(1 - 4x + 4x^2\right) - \frac{x}{2} c_1^{\,2}\left(x^2 - 2x^3 + x^4\right) - c_1 x\left(x - x^2\right)\right] dx$

$$= \frac{c_1^{\,2}}{2}\left(1 - 2 + \frac{4}{3}\right) - \frac{c_1^{\,2}}{2}\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) - c_1\left(\frac{1}{3} - \frac{1}{4}\right) \quad = \frac{19}{120} c_1^{\,2} - \frac{c_1}{12}$$

$$\frac{\partial I}{\partial c_1} = 0 \;\Rightarrow\; \frac{19}{60} c_1 - \frac{1}{12} = 0 \;\Rightarrow\; c_1 = 0.263 \;\Rightarrow\; y(1) = 0.263x(1 - x)$$

(2)Two-term approx

$$y = x(1 - x)(c_1 + c_2 x) = c_1(x - x^2) + c_2(x^2 - x^3)$$

Then $y' = c_1(1-2x) + c_2(2x-3x^2)$

$$I(c_1,c_2) = \int_0^1 [\frac{1}{2}\left\{c_1{}^2\left(1-4x+4x^2\right) + 2c_1c_2\left(2x-7x^2+6x^3\right)\right.$$

$$\left. + c_3{}^2(4x^2 - 12x^3 + 9x^4)\right\} - \frac{1}{2}\left\{c_1{}^2\left(x^3-2x^4+x^5\right) + 2c_1c_2\left(x^4-2x^5+x^6\right)\right.$$

$$\left. + c_2{}^2(x^5 - 2x^6 + x^7)\right\} - \left\{c_1\left(x^2-x^3\right) + c_2\left(x^3-x^4\right)\right\}]dx$$

$$= \frac{c_1{}^2}{2}\left(1-2+\frac{4}{3}-\frac{1}{4}+\frac{2}{5}-\frac{1}{6}\right) + c_1c_2\left(1-\frac{7}{3}+\frac{3}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}\right)$$

$$-\frac{c_1{}^2}{2}\left(\frac{4}{3}-3+\frac{9}{5}-\frac{1}{6}+\frac{2}{7}-\frac{9}{8}\right) - \frac{c_1}{12} - \frac{c_2}{20}$$

$$= \frac{19}{120}c_1{}^2 + \frac{11}{70}c_1c_2 + \frac{107}{1680}c_2{}^2 - \frac{c_1}{12} - \frac{c_2}{20}$$

$$\frac{\partial I}{\partial c_1} = 0 \qquad \Rightarrow \quad \frac{19}{60}c_1 + \frac{11}{70}c_2 = \frac{1}{12}$$

$$\frac{\partial I}{\partial c_2} = 0 \qquad \Rightarrow \quad \frac{11}{70}c_1 + \frac{109}{840}c_2 = \frac{1}{20}$$

0.317 c_1 + 0.127 c_2 = 0.05 $\Rightarrow$ c_1 = 0.177 , c_2 = 0.173

$$\Rightarrow \quad y(2) = (0.177x - 0.173x^2)(1-x)$$

(It is noted that the deviation between the successive approxs. y(1) and y(2) is found to be smaller in magnitude than 0.009 over (0,1))