COMPLEX ANALYSIS SINGULARITY

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ZERO OF AN ANALYTIC FUNCTION-:

- *A ZERO OF AN ANALYTIC FUNCTION $f(z)$ IS A VALUE OF Z SUCH THAT $f(z) = 0$*
- *Suppose $f(z)$ is analytic in a domain D and a is any point in D. Then $f(z)$ can be expanded as a Taylor's series about $z=a$ in the form*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where,

$$a_n = f^{(n)}(a)$$

Singular Points :-

- A singularity of a function is the point at which the function ceases be analytic.

Example $f(z) = 1/(z-2)$

Then $z=2$ is a singularity of $f(z)$

Types of singularity:-

- Isolated Singularity
- Removable singularity
- Pole
- Essential singularity
- Non Isolated Singularity

Isolated Singularity:-

A point $z=a$ is said to be isolated singularity of $f(z)$ if

- $F(z)$ is not analytic at $z=a$
- $F(z)$ is analytic in the deleted nbd of $z=a$, i.e. there exists a nbd of $z=a$ containing no other singularity

Example

- The function $f(z) = 1/z$ is analytic everywhere except at $z=0$, therefore $z=0$ is an isolated singularity.

<u>*Non Isolated Singularity:-*</u>

A point z=a is called a non isolated singularity of f(z) if z=a is a singularity and every deleted nbd of z=a contains at least one singularity of f(z)

Example:-

- F(z)=tan(1/z)

<u>Definition:-</u>

*L*et z=a be an isolated singularity of a function f(z),then by definition ;there exists a deleted neighbourhood

$0<|z-a|<r$

in which f(z) is analytic.Hence,if z be any point of this neighbourhood,then by Laurent expansion,

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty} b_n\,(z-a)^{-n}$$

Removable Singularity:-

- If the principal part of f(z) contains no term i.e. b_n =0

or all n then the singularity z=a is called removable singularity of f(z).In this case

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Example:-

$$f(z)=\sin z / z$$

Pole:-

- If the principal part contains a finite number of terms, say m then the singularity $z=a$ is called a pole of order m of $f(z)$.
- *A pole of order one is called a* **simple pole.**

An Alternate definition:-

A function $f(z)$ *is said to have pole of order n if it is expressible as*

$$f(z) = \Phi(z)/(z-a)^n,$$

where $\Phi(z)$ *is analytic and* $\Phi(a) \neq 0$

Zeros are isolated:-

- ***Theorem:-***

Let *f(z) be analytic in a domain D.Then unless f(z) is identically zero,there exists a neighbourhood of each point in D through out which the function has no zero, except possibly at the point itself.In other words the zero of an analytic function are isolated.*

- **Proof:-**

Let z=a be a zero of order m of an analytic function f(z).Then we may write $f(z) = (z-a)^m \Phi(z)$, *where* $\Phi(z)$ is analytic and $\Phi(a) \neq 0$.

now there exists no other point in the deleted neighbourhood $|z-a|<r$ at which $f(z)=0$.

Hence the zero z=a is an isolated singularity.

Therefore the zeros of f(z) are isolated singularity.

Poles are isolated:-

Let z=a be a pole of order m of an analytic function *f(z), then*
1 / f(z) is analytic and has a zero of order m at z=a.

- ***Zeros are isolated therefore poles are also isolated***.

Limiting point of zeros:-

- Let *f(z) be an analytic function in a simply connected region D.*
  Let $a_{1,} a_{2,} a_{3,} a_{4,} - - - - - - - - - - a_n \; - - - -$
  be a sequence of zeros having a as its limit point, a being the interior point of D. then either f(z) vanishes identically or else has an isolated essential singularity at z=a.
- *Limit point of zeros is an isolated essential singularity.*

Limit point of poles:-

- Suppose z=a is a limit point of the sequence of poles of an analytic function *f(z). Then every neighbourhood of the point z=a containing poles of the given function. Therefore the point z=a is a singularity of f(z). Such a singularity is called non isolated essential singularity.*
- *Limit point of poles is a non isolated essential singularity.*