# COMPLEX ANALYSIS SINGULARITY

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## **ZERO OF AN ANALYTIC FUNCTION-:**

- A ZERO OF AN ANALYTIC FUNCTION f(z) IS A VALUE OF Z SUCH THAT f(z) = 0
- Suppose f(z) is analytic in a domain D and a is any point in D. Then f(z) can be expanded as a Taylor's series about z=a in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where,

 $a_n = f^{(n)}(a)$ 

# Singular Points :-

- A singularity of a function is the point at which the function ceases be analytic.
- Example f(z) = 1/(z-2)Then z=2 is a singularity of f(z)

### <u>Types of singularity:-</u>

- Isolated Singularity
- Removable singularity
- Pole
- Essential singularity
- Non Isolated Singularity

# **Isolated Singularity:-**

- A point z=a is said to be isolated singularity of f(z) if
- F(z) is not analytic at z=a
- F(z) is analytic in the deleted nbd of z=a,i.e. there exists a nbd of z=a containing no other singularity

### Example

• The function f(z) = 1/z is analytic everywhere except at z=0, therefore z=0 is an isolated singularity.

# Non Isolated Singularity:-

A point z=a is called a non isolated singularity of f(z) if z=a is a singularity and every deleted nbd of z=a contains at least one singularity of f(z)

Example:-

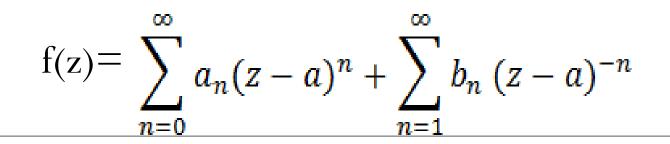
•  $F(z) = \tan(1/z)$ 

#### <u>Definition:-</u>

Let z=a be an isolated singularity of a function f(z), then by definition ; there exists a deleted neighbourhood

 $0 \le |z - a| \le r$ 

in which f(z) is analytic. Hence, if z be any point of this neighbourhood, then by Laurent expansion,



### <u>Removable Singularity:-</u>

• If the principal part of f(z) contains no term i.e.  $b_n=0$ 

or all n then the singularity z=a is called removable singularity of f(z). In this case

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Example:-

$$f(z) = \sin z / z$$

### Pole:-

If the principal part contains a finite number of terms, say m then the singularity z=a is called a pole of order m of *f(z)*.
A pole of order one is called a simple pole.

#### An Alternate definition:-

A function f(z) is said to have pole of order n if it is expressible as

 $f(z) = \Phi(z)/(z-a)^n$ , where  $\Phi(z)$  is analytic and  $\Phi(a) \neq 0$ 

## Zeros are isolated:-

#### **D** <u>Theorem:-</u>

Let f(z) be analytic in a domain D. Then unless f(z) is identically zero, there exists a neighbourhood of each point in D through out which the function has no zero, except possibly at the point itself. In other words the zero of an analytic function are isolated.

#### Proof:-

Let z=a be a zero of order m of an analytic function f(z). Then we may write  $f(z) = (z-a)^{n} \Phi(z)$ , where  $\Phi(z)$  is analytic and  $\Phi(a) \neq 0$ . now there exists no other point in the deleted neighbourhood |z-a| < r at which f(z)=0.

*Hence the zero z*=*a is an isolated singularity.* 

Therefore the zeros of f(z) are isolated singularity.

## Poles are isolated:-

Let z=a be a pole of order m of an analytic function f(z), then 1/f(z) is analytic and has a zero of order m at z=a.

**Zeros** are isolated therefore poles are also isolated.

### Limiting point of zeros:-

Let f(z) be an analytic function in a simply connected region D. Let  $a_{1,a_{2,a_{3,a_{4,}}} - - - - - - a_n - - - - a_n$ 

be a sequence of zeros having a as its limit point, a being the interior point of D. then either f(z) vanishes identically or else has an isolated essential singularity at z=a.

□ Limit point of zeros is an isolated essential singularity.

## Limit point of poles:-

- Suppose z=a is a limit point of the sequence of poles of an analytic function f(z). Then every neighbourhood of the point z=a containing poles of the given function. Therefore the point z=a is a singularity of f(z). Such a singularity is called non isolated essential singularity.
- Limit point of poles is a non isolated essential singularity.